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Elementary Equivalence of Endomorphism Rings of Abelian p-Groups

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Contents

Introduction

In this paper, we consider elementary properties (i.e., properties which are definable in the first order language) of endomorphism rings of Abelian p-groups.

The first result on relationship of elementary properties of some models with elementary properties of derivative models was proved by A. I. Maltsev in 1961 in [21]. He proved that the groups $G_n(K)$ and $G_m(L)$ (where $G = \mathrm{GL}, \mathrm{SL}, \mathrm{PGL}, \mathrm{PSL}$, $n, m \geq 3$, K, L are fields of characteristic 0) are elementarily equivalent if and only if $m = n$ and the fields K and L are elementarily equivalent.

This theory was continued in 1992 when with the help of the construction of ultraproduct and the isomorphism theorem [10] C. I. Beidar and A. V. Mikhalev in [3] formulated a general approach to problems of elementary equivalence of various algebraic structures, and generalized Maltsev's theorem for the case where K and L are skewfields and associative rings.

In 1998–2001 E. I. Bunina continued to study some problems of this type (see [5, 6, 7, 8]). She generalized the results of A. I. Maltsev for unitary linear groups over skewfields and associative rings with involution, and also for Chevalley groups over fields.

In 2000 V. Tolstykh in [28] studied a relationship between second order properties of skewfields and first order properties of automorphism groups of infinite-dimensional linear spaces over them. In 2003 (see [9]) the authors studied a relationship between second order properties of associative rings and first order properties of categories of modules, endomorphism rings, automorphism groups, and projective spaces of infinite rank over these rings.

In this paper, we study a relationship between second order properties of Abelian p-groups and first order properties of their endomorphism rings.

The first section includes some basic notions from the set theory and model theory: definitions of first order and second order languages, models of a language, deducibility, interpretability, basic notions of set theory, which will be needed in next sections.

The second section contains all notions and statements about Abelian groups which will be needed for our future constructions. We have taken them mainly from [15].

In the third section, we show how to extend the results of S. Shelah from [24] on interpreting the set theory in a category for the case of the endomorphism ring of some special Abelian p-group, which is a direct sum of cyclic groups of the same order.

In Sec. 4, we describe the second order group language $\mathcal{L}_2$, and also its restriction $\mathcal{L}_2^{\varkappa}$ by some cardinal number $\varkappa$, and then in Sec. 4.2 we introduce the *expressible rank* $r_{\exp}$ of an Abelian group A, represented as the direct sum $D \oplus G$ of its divisible and reduced components, as the maximum of the powers of the group D and some basic subgroup B of A, i.e., $r_{\exp} = \max(r(D), r(B))$. In Sec. 4.2, we also formulate the main theorem of this work.

Theorem 1. *For any infinite p-groups A_1 and A_2 elementary equivalence of endomorphism rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *implies coincidence of the second order theories* $\mathrm{Th}_2^{r_{\exp}(A_1)}(A_1)$ *and* $\mathrm{Th}_2^{r_{\exp}(A_2)}(A_2)$ *of the groups A_1 and A_2, bounded by the cardinal numbers* $r_{\exp}(A_1)$ *and* $r_{\exp}(A_2)$*, respectively.*

Note that $r_{\exp}(A) = |A|$ in all cases except the case where $|D| < |G|$, any basic subgroup of A is countable, and the group G itself is uncountable. In this case, $r_{\exp}(A) = \omega$.

In Sec. 4.3, we prove two “inverse implications” of the main theorem.

Theorem 2. *For any Abelian groups A_1 and A_2, if the groups A_1 and A_2 are equivalent in the second order logic $\mathcal{L}_2$, then the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *are elementarily equivalent.*

Theorem 3. *If Abelian groups A_1 and A_2 are reduced and their basic subgroups are countable, then* $\mathrm{Th}_2^{\omega}(A_1) = \mathrm{Th}_2^{\omega}(A_2)$ *implies elementary equivalence of the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$.

Therefore for all Abelian groups, except the case where $A = D \oplus G$, $D \neq 0$, $|D| < |G|$, and $|G| > \omega$, a basic subgroup in A is countable, and elementary equivalence of the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ is equivalent to

$$\mathrm{Th}_2^{r_{\exp}(A_1)}(A_1) = \mathrm{Th}_2^{r_{\exp}(A_2)}(A_2).$$

In Sec. 4.4, we divide the proof of the main theorem into three cases:

1. A_1 and A_2 are bounded;
2. $A_1 = D_1 \oplus G_1$, $A_2 = D_2 \oplus G_2$, D_1 and D_2 are divisible, G_1 and G_2 are bounded;
3. A_1 and A_2 have unbounded basic subgroups.

In Secs. 5–7, these three cases are under consideration.

In Sec. 8, we prove the main theorem, combining all three cases in one proof.

1 Basic Notions from Model Theory

1.1 First Order Languages

A *first order language* $\mathcal{L}$ is a collection of symbols. It consists of (1) parentheses (,); (2) connectives $\wedge$ (“and”) and $\neg$ (“not”); (3) the quantifier $\forall$ (for all); (4) the binary relation symbol $=$ (identity); (5) a countable set of variables x_i; (6) a finite or countable set of relation symbols Q_i^n $(n \geq 1)$; (7) a finite or countable set of function symbols F_i^n $(n \geq 1)$; (8) a finite or countable set of constant symbols c_i.

Now we introduce the most important for us examples of first order languages: the group language $\mathcal{L}_G$ and the ring language $\mathcal{L}_R$.

We assume that in the group language there are neither function nor constant symbols, and there is a unique 3-place relation symbol Q^3, which corresponds to multiplication. Instead of $Q^3(x_1, x_2, x_3)$ we shall write $x_1 = x_2 \cdot x_3$, or $x_1 = x_2 x_3$.

For the ring language we shall also suppose that there are neither function nor constant symbols, and there are two relation symbols: a 3-place symbol of multiplication Q_1^3 (instead of $Q_1^3(x_1, x_2, x_3)$ we shall write $x_1 = x_2 \cdot x_3$, or $x_1 = x_2 x_3$) and a 3-place symbol of addition Q_2^3 (instead of $Q_2^3(x_1, x_2, x_3)$ we shall write $x_1 = x_2 + x_3$).

A *symbol-string* is defined as follows: (1) every symbol α of the language $\mathcal{L}$ is a *symbol-string*; (2) if σ and ρ are symbol-strings, than $\sigma\rho$ is a symbol-string. A *designating symbol-string σ for a symbol-string ρ* is the symbol-string $\sigma := \rho$, or $\rho := \sigma$ (*σ is a designation for ρ*). If a symbol-string ρ is a part of a symbol-string σ, staying in one of the three following positions: $\ldots\rho$, $\rho\ldots$, $\ldots\rho\ldots$, then ρ is an *occurrence in σ*.

Some symbol-strings constructed from the symbols of the language $\mathcal{L}$ are called *terms* and *formulas* of this language.

Terms are defined as follows:

1. a variable is a term;
2. a constant symbol is a term;
3. if F^n is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then $F^n(t_1, \ldots, t_n)$ is a term;
4. a symbol-string is a term only if it can be shown to be a term by a finite number of applications of (1)–(3).

In the cases of the languages $\mathcal{L}_G$ and $\mathcal{L}_R$, terms have the form x_i.

The *elementary formulas* of the language $\mathcal{L}$ are symbol-strings of the form given below:

1. if t_1 and t_2 are terms of the language $\mathcal{L}$, then $t_1 = t_2$ is an elementary formula;
2. if Q^n is an n-place relation symbol and $t_1, \ldots, t_n$ are terms, then the symbol-string $Q^n(t_1, \ldots, t_n)$ is an elementary formula.

For the language $\mathcal{L}_G$ the elementary formulas have the form $x_i = x_j$ and $x_i = x_j \cdot x_k$, and for the language $\mathcal{L}_R$ they have the form $x_i = x_j$, $x_i = x_j \cdot x_k$, and $x_i = x_j + x_k$.

Finally, the *formulas* of the language $\mathcal{L}$ are defined as follows:

1. an elementary formula is a formula;
2. if φ and ψ are formulas and x is a variable, then $(\neg\varphi)$, $(\varphi \wedge \psi)$, and $(\forall x\, \varphi)$ are formulas;
3. a symbol-string is a formula only if it can be shown to be a formula by a finite number of applications of (1)–(2).

Let us introduce the following abbreviations:

$(\varphi \vee \psi)$ stands for $(\neg((\neg\varphi) \wedge (\neg\psi)))$;

$(\varphi \Rightarrow \psi)$ stands for $((\neg\varphi) \vee \psi)$;

$(\varphi \Leftrightarrow \psi)$ stands for $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$;

$(\exists x\, \varphi)$ is an abbreviation for $(\neg(\forall x\, (\neg\varphi)))$;

$\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$ stands for $(\varphi_1 \vee (\varphi_2 \vee \cdots \vee \varphi_n))$;

$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ stands for $(\varphi_1 \wedge (\varphi_2 \wedge \cdots \wedge \varphi_n))$;

$\forall x_1 \ldots \forall x_n \varphi$ stands for $(\forall x_1) \ldots (\forall x_n)\varphi$;

$\exists x_1 \ldots \exists x_n \varphi$ stands for $(\exists x_1) \ldots (\exists x_n)\varphi$.

Let us introduce the notions of *free* and *bound* occurrences of a variable in a formula.

1. All occurrences of all variables in elementary formulas are free occurrences.
2. Every free (bound) occurrence of a variable x in a formula φ is a free (bound) occurrence of the variable x in the formulas $(\neg\varphi)$, $(\varphi \wedge \psi)$, and $(\psi \wedge \varphi)$.
3. For any occurrence of a variable x in a formula φ, the occurrence of the variable x in the formula $\forall x\, \varphi$ is bound. If an occurrence of a variable x in a formula φ is free (bound), then the occurrence of x in $\forall x'\, \varphi$ is free (bound).

Therefore, one variable can have free and bound occurrences in the same formula. A variable is called a *free* (*bound*) *variable* in a given formula if there exist free (bound) occurrences of this variable in this formula. Thus a variable can be free and bound in the same time.

A *sentence* is a formula with no free variables.

Let φ be a formula, t be a term, and x be a variable. The *substitution* of a term t into the formula φ for the variable x is the formula $\varphi(t \mid x)$, obtained by replacing every free occurrence of the variable x in φ by the term t. The substitution $\varphi(t \mid x)$ is called *admissible* if for every variable x' occurring in the term t no free occurrence of x in φ is a part of a subformula $\forall x'\, \psi(x')$ or $\exists x'\, \psi(x')$ of the formula φ.

For example, in the case of the group language $\mathcal{L}_G$ terms are variables. If we have a formula $\forall x_1\,(x_2 = x_1)$, then the substitution of the term x_1 for x_2 is not admissible, and the substitution of the term x_3 for x_2 is admissible.

For the formula $\forall x_1\,(x_2 = x_1 \cdot x_3)$ the substitution $x_1 \mid x_2$ is not admissible, and the substitution $x_3 \mid x_2$ is admissible.

Now let us introduce the following convention of notation: we use $t(x_1, \ldots, x_n)$ to denote a term t whose variables form a subset of $\{x_1, \ldots, x_n\}$. Similarly, we use $\varphi(x_1, \ldots, x_n)$ to denote a formula whose *free* variables form a subset of $\{x_1, \ldots, x_n\}$.

We need *logical axioms* and *rules of inference* to construct a formal system. Logical axioms are cited below.

Purely logical axioms.

1. $\varphi \Rightarrow (\psi \Rightarrow \varphi)$.

2. $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$.

3. $(\neg\varphi \Rightarrow \neg\psi) \Rightarrow ((\neg\psi \Rightarrow \varphi) \Rightarrow \psi)$.

4. $\forall x \varphi(x) \Rightarrow \varphi(t|x)$ if t is a term such that the substitution $t \mid x$ is admissible.

5. $(\forall x(\psi \Rightarrow \varphi(x))) \Rightarrow (\psi \Rightarrow (\forall x \varphi))$ if ψ does not contain any free occurrences of x.

Identity axioms.

1. $x = x$.

2. $y = z \Rightarrow t(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) = t(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$.

3. $y = z \Rightarrow (\varphi(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \Leftrightarrow \varphi(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n))$, where $x_1, \ldots, x_n, y, z$ are variables, t is a term, and $\varphi(x_1, \ldots, x_n)$ is an elementary formula.

There are two inference rules.

1. The rule of detachment (modus ponens or MP): from φ and $\varphi \Rightarrow \psi$ infer ψ.

2. The rule of generalization: from φ infer $\forall x\, \varphi$.

Let Σ be a collection of formulas and ψ be a formula of the language $\mathcal{L}$. A sequence $(\varphi_1, \ldots, \varphi_n)$ of formulas of the language $\mathcal{L}$ is called a *deduction of the formula ψ from the collection* Σ if $\varphi_n = \psi$ and for any $1 \leq i \leq n$ one of the following conditions is fulfilled:

1. φ_i belongs to Σ or is a logical axiom;

2. there exist $1 \leq k < j < i$ such that φ_j is $(\varphi_k \Rightarrow \varphi_i)$, i.e., φ_i is obtained from φ_k and $\varphi_k \Rightarrow \varphi_i$ by the inference rule MP;

3. there exists $1 \leq j < i$ such that φ_i is $\forall x\, \varphi_j$, where x is not a free variable of any formula from Σ.

Denote this deduction by $(\varphi_1, \ldots, \varphi_n)\colon \Sigma \vdash \psi$.

If there exists a deduction $(\varphi_1, \ldots, \varphi_n)\colon \Sigma \vdash \psi$, then the formula ψ is called *deducible in the language $\mathcal{L}$ from the set* Σ, and the deduction $(\varphi_1, \ldots, \varphi_n)$ is called a *proof of ψ*.

A (*first order*) *theory* T in the language $\mathcal{L}$ is some set of sentences of the language $\mathcal{L}$. A *set of axioms* of the theory T is any set of sentences which has the same corollaries as T.

1.2 Theory of Classes and Sets NBG

The set theory of von Neumann, Bernays, and Gödel NBG (see [22]), which will be a base for all our constructions, has one relation symbol P^2, which denotes a 2-place relation, no function, and no constant symbols. We shall use Latin letters X, Y, and Z with subscripts as variables of this system. We also introduce the abbreviations $X \in Y$ for $P(X, Y)$ and $X \notin Y$ for $\neg P(X, Y)$. The sign $\in$ can be interpreted as the symbol of belonging.

The formula $X \subseteq Y$ is an abbreviation for the formula $\forall Z\,(Z \in X \Rightarrow Z \in Y)$ (*inclusion*), $X \subset Y$ is an abbreviation for $X \subseteq Y \wedge X \neq Y$ (*proper inclusion*).

Objects of the theory NBG are called *classes*. A class is called a *set* if it is an element of some class. A class which is not a set is called a *proper class*. We introduce small Latin letters x, y, and z with subscripts as special variables bounded by sets. This means that the formula $\forall x\, A(x)$ is an abbreviation for $\forall X\,(X \text{ is a set} \Rightarrow A(X))$, and it has the sense "$A$ is true for all sets", and $\exists x\, A(x)$ is an abbreviation for $\exists X\,(X \text{ is a set} \wedge A(X))$, and it has the sense "$A$ is true for some set."

A1 (*the extensionality axiom*). $\forall X\,\forall Y\,(X = Y \Leftrightarrow \forall Z\,(Z \in X \Leftrightarrow Z \in Y))$. Intuitively, $X = Y$ if and only if X and Y have the same elements.

A2 (*the pair axiom*). $\forall x\,\forall y\,\exists z\,\forall u\,(u \in z \Leftrightarrow u = x \vee u = y)$, i.e., for all sets x and y there exists a set z such that x and y are the only elements of z.

A3 (*the empty set axiom*). $\exists x\,\forall y\,\neg(y \in x)$, i.e., there exists a set which does not contain any elements.

Axioms **A1** and **A3** imply that this set is unique, i.e., we can introduce a constant symbol $\varnothing$ (or 0), with the condition $\forall y\,(y \notin \varnothing)$.

Also we can introduce a new function symbol $f(x, y)$ for the pair, and write it in the form $\{x, y\}$. Further, let $\{x\} = \{x, x\}$. The set $\langle x, y\rangle \equiv \{\{x, \{x, y\}\}$ is called the *ordered pair* of sets x and y.

Proposition 1. $\vdash \forall x\,\forall y\,\forall u\,\forall v\,(\langle x, y\rangle = \langle u, v\rangle \Rightarrow x = u \wedge y = v)$.

In the same way we can introduce *ordered triplets of sets, ordered quadruplets of sets*, and so on.

AS4 (*the axiom scheme of existence of classes*). Let

$$\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$$

be a formula. We shall call this formula *predicative* if only variables for sets are bound in it (i.e., if it can be transferred to this form with the help of abbreviations). For every predicative formula $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$

$$\exists Z\,\forall x_1 \ldots \forall x_n\,(\langle x_1, \ldots, x_n\rangle \in Z \Leftrightarrow \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)).$$

The class Z which exists by the axiom scheme **AS4** will be denoted by

$$\{x_1, \ldots, x_n \mid \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)\}.$$

Now, by the axiom scheme **AS4**, we can define for arbitrary classes X and Y the following derivative classes:

$X \cap Y \equiv \{u \mid u \in X \wedge u \in Y\}$ (*the intersection of classes X and Y*);

$X \cup Y \equiv \{u \mid u \in X \vee u \in Y\}$ (*the union of classes X and Y*);

$\bar{X} \equiv \{u \mid u \notin X\}$ (*the complement of a class X*);

$V \equiv \{u \mid u = u\}$ (*the universal class*);

$X \setminus Y \equiv \{u \mid u \in X \wedge u \notin Y\}$ (*the difference of classes X and Y*);

$\mathrm{Dom}(X) \equiv \{u \mid \exists v\, (\langle u, v\rangle \in X)\}$ (*the domain of a class X*);

$\mathrm{Rng}(X) \equiv \{u \mid \exists v(\langle v, u\rangle \in X)\}$ (*the image of a class X*);

$X \times Y \equiv \{u \mid \exists x\, \exists y\, (u = \langle x, y\rangle \wedge x \in X \wedge y \in Y)\}$ (*the Cartesian product of classes X and Y*);

$\mathcal{P}(X) \equiv \{u \mid u \subseteq X\}$ (*the class of all subsets of a class X*);

$\cup X \equiv \{u \mid \exists v\, (u \in v \wedge v \in X)\}$ (*the union of all elements of a class X*).

Introduce now other axioms.

A5 (*the union axiom*). $\forall x\, \exists y\, \forall u\, (u \in y \Leftrightarrow \exists v\, (u \in v \wedge v \in x))$.

This axiom states that the union $\cup x$ of all elements of a set x is also a set.

A6 (*the power set axiom*). $\forall x\, \exists y\, \forall u\, (u \in y \Leftrightarrow u \subseteq x)$.

This axiom states that the class of all subsets of a set x is a set, which will be called the *power set of x*.

A7 (*the separation axiom*). $\forall x\, \forall Y\, \exists z\, \forall u\, (u \in z \Leftrightarrow u \in x \wedge u \in Y)$.

This axiom states that the intersection of a class and a set is a set.

Denote the class $X \times X$ by X^2, the class $X \times X \times X$ by X^3, and so on. Let the formula $\mathrm{Rel}(X)$ be an abbreviation for the formula $X \subseteq V^2$ (*X is a relation*), $\mathrm{Un}(X)$ be an abbreviation for the formula $\forall x\, \forall y\, \forall z\, (\langle x, y\rangle \in X \wedge \langle x, z\rangle \in X \Rightarrow y = z)$ (*X is functional*), and $\mathrm{Fnc}(X)$ be an abbreviation for $X \subseteq V^2 \wedge \mathrm{Un}(X)$ (*X is a function*).

A8 (*the replacement axiom*). $\forall X\, \forall x\, (\mathrm{Un}(X) \Rightarrow \exists y\, \forall u\, (u \in y \Leftrightarrow \exists v\, (\langle v, u\rangle \in X \wedge v \in x)))$.

This axiom states that if the class X is functional, then the class of second components of pairs from X such that the first component belongs to x is a set. The following axiom postulates existence of an infinite set.

A9 (*the infinity axiom*). $\exists x\, (0 \in x \wedge \forall u\, (u \in x \Rightarrow u \cup \{u\} \in x))$. It is clear that for such a set x we have $\{0\} \in x$, $\{0, \{0\}\} \in x$, $\{0, \{0\}, \{0, \{0\}\}\} \in x$,... If we now set $1 := \{0\}$, $2 := \{0, 1\}$,..., $n := \{0, 1, \ldots, n-1\}$, then for every integer $n \geq 0$ the condition $n \in x$ is fulfilled and $0 \neq 1$, $0 \neq 2$, $1 \neq 2$,...

A10 (*the regularity axiom*). $\forall X\, (X \neq \varnothing \Rightarrow \exists x \in X\, (x \cap X = \varnothing))$.

This axiom states that every nonempty set is disjoint from one of its elements.

A11 (*the axiom of choice AC*). For every set x there exists a mapping f such that for every nonempty subset $y \subseteq x$ we have $f(y) \in y$ (this mapping is called a *choice* mapping for x).

The list of axioms of the theory NBG is finished.

A class P is called *ordered by a binary relation $\leq$ on P*, if the following conditions hold

1. $\forall p \in P\, (p \leq p)$;
2. $\forall p, q \in P\, (p \leq q \wedge q \leq p \Rightarrow p = q)$;
3. $\forall p, q, r \in P\, (p \leq q \wedge q \leq r \Rightarrow p \leq r)$.

If, in addition,

4. $\forall p, q \in P\, (p \leq q \vee q \leq p)$,

then the relation $\leq$ is called a *linear order* on the class P.

An ordered class P is called *well-ordered* if

5. $\forall q\,(\varnothing \neq q \subseteq P \Rightarrow \exists x \in q\,(\forall y \in q\,(x \leq y)))$, i.e., every nonempty subset of the class P has the smallest element.

A class S is called *transitive* if $\forall x\,(x \in S \Rightarrow x \subseteq S)$.

A class (a set) S is called an *ordinal* (an *ordinal number*) if S is transitive and well-ordered by the relation $\in \cup =$ on S.

Ordinal numbers are usually denoted by Greek letters α, β, γ, and so on. The class of all ordinal numbers is denoted by On. The natural ordering of the class of ordinal numbers is the relation $\alpha \leq \beta := \alpha = \beta \vee \alpha \in \beta$. The class On is transitive and linearly ordered by the relation $\leq$.

There are some simple assertions about ordinal numbers:

1. if α is an ordinal number, a is a set, and $a \in \alpha$, then a is an ordinal number;

2. $\alpha + 1 \equiv \alpha \cup \{\alpha\}$ is the smallest ordinal number that is greater than α;

3. every nonempty set of ordinal numbers has the smallest element.

Therefore the ordered class On is well-ordered. Thus On is an ordinal.

An ordinal number α is called a *successor* if $\alpha = \beta + 1$ for some ordinal number β. In the opposite case α is called a *limit ordinal number*.

The smallest (in the class On) nonzero limit ordinal is denoted by ω. Ordinals which are smaller than ω are called *natural numbers*.

Classes F which are functions with domains equal to ω are called *infinite sequences*. Functions with domains equal to $n \in \omega$ are called *finite sequences*.

Sets a and b are called *equivalent* ($a \sim b$) if there exists a bijective function $u\colon a \to b$.

An ordinal number α is called a *cardinal* if for every ordinal number β the conditions $\beta \leq \alpha$ and $\beta \sim \alpha$ imply $\beta = \alpha$. The class of all cardinal numbers is denoted by Cn. The class Cn with the order induced from the class On is well-ordered.

The axiom of choice implies that for every set a there exists a unique cardinal number α such that $a \sim \alpha$. This number α is called the *power of the set* a (denoted by $|a|$, or $\operatorname{card} a$). A set of power ω is called *countable*. A set of power $n \in \omega$ is called *finite*. A set is called *infinite* if it is not finite. A set is called *uncountable* if it is neither countable, nor finite. The cardinal number $c := |\mathcal{P}(\omega)|$ is called the power of *continuum*.

To denote cardinals we shall use small Greek letters (as in the case of ordinals): ξth infinite cardinal will be denoted by ω_ξ (i.e., the cardinal number ω will also be denoted by ω_0).

A set X is said to be *cofinal* in α if $X \subset \alpha$ and $\alpha = \cup X$. The cofinality of α, written $\mathrm{cf}\alpha$, is the least cardinal β such that a set of power β is cofinal in α.

A cardinal $\varkappa$ is said to be *regular* if $\mathrm{cf}\varkappa = \varkappa$, i.e., for every ordinal number β for which there exists a function $f\colon \beta \to \varkappa$ such that $\cup \operatorname{rng} f = \varkappa$ the inequality $\varkappa \leq \beta$ holds, where $\cup \operatorname{rng} f = \varkappa$ means that for every $y \in \varkappa$ there exists $x \in \beta$ such that $y < f(x)$. A cardinal $\varkappa$ is said to be *singular* if it is not regular.

The continuum hypothesis states that $|\mathcal{P}(\omega)| = \omega_1$, i.e., the power of continuum is the smallest uncountable cardinal number.

We shall assume the continuum hypothesis if we need it.

1.3 Models, Satisfaction, and Elementary Equivalence

We now suppose that all our constructions are made in the theory NBG.

A *model of a first order language* $\mathcal{L}$ is a pair $\mathcal{U} = \langle A, I\rangle$ consisting of a universe A (i.e., some class or set of the theory NBG) and some correspondence I that assigns to every relation symbol Q^n some n-place relation $R \subset A^n$ on A, to every function symbol F^n some m-place function $G\colon A^m \to A$, and to every constant symbol c some element of A.

A simple example of a model of the group language is the set $1 := \{\varnothing\} = \{0\}$, where $I(Q^2) = \{\langle 0,0,0\rangle\}$. Another simple example of a model of the group language is the set $2 := \{\varnothing, \{\varnothing\}\} = \{0,1\}$, where $I(Q^2) = \{\langle 0,0,0\rangle, \langle 0,1,1\rangle, \langle 1,0,1\rangle, \langle 1,1,0\rangle\}$.

The *power* of a model $\mathcal{U} = \langle A, I\rangle$ is the cardinal number $|A|$ (if the universe A is a set). For all models which will be considered in this paper, the universe A is a set. A model $\mathcal{U}$ is called finite, countable, or uncountable if $|A|$ is finite, countable, or uncountable, respectively.

Models $\mathcal{U}$ and $\mathcal{U}'$ of a language $\mathcal{L}$ are called *isomorphic*, if there exists a bijective mapping f of the set (universe) A onto the set A' satisfying the following conditions:

1. for each n-place relation symbol Q^n and any $a_1, \ldots, a_n$ from A

$$\langle a_1, \ldots, a_n\rangle \in I(Q^n) \text{ if and only if } \langle f(a_1), \ldots, f(a_n)\rangle \in I'(Q^n);$$

2. for each m-place function symbol F^m of the language $\mathcal{L}$ and any $a_1, \ldots, a_m \in A$

$$f(I(F^m)(\langle a_1, \ldots, a_m\rangle)) = I'(F^m)(\langle f(a_1), \ldots, f(a_m)\rangle);$$

3. for each constant symbol c of the language $\mathcal{L}$

$$f(I(c)) = I'(c).$$

Every mapping f satisfying these conditions is called an *isomorphism of the model* $\mathcal{U}$ *onto the model* $\mathcal{U}'$ or an *isomorphism between the models* $\mathcal{U}$ *and* $\mathcal{U}'$. The fact that f is an isomorphism of the model $\mathcal{U}$ onto the model $\mathcal{U}'$ will be denoted by $f\colon \mathcal{U} \cong \mathcal{U}'$, and the formula $\mathcal{U} \cong \mathcal{U}'$ means that the models $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic. For convenience we use $\cong$ to denote the isomorphism relation between models.

Indeed, unless we wish to consider the particular structure of each element of A or A', for all practical purposes $\mathcal{U}$ and $\mathcal{U}'$ are the same if they are isomorphic.

Now we shall give a formal definition of satisfiability. Let φ be an arbitrary formula of a language $\mathcal{L}$, let all its variables, free and bound, be contained in the set $x_1, \ldots, x_q$, and let $a_1, \ldots, a_q$ be an arbitrary sequence of elements of the set A. We define the predicate

φ *is true on the sequence* $a_1, \ldots, a_q$ *in the model* $\mathcal{U}$, or $a_1, \ldots, a_q$ *satisfies the formula* φ *in* $\mathcal{U}$.

The definition proceeds in three stages. Let $\mathcal{U}$ be a fixed model for $\mathcal{L}$.

1. The value of a term $t(x_1, \ldots, x_q)$ at $a_1, \ldots, a_q$ is defined as follows (we let $t[a_1, \ldots, a_q]$ denote this value):

1. if t is a variable x_i, then $t[a_1, \ldots, a_q] = a_i$;

2. if t is a constant symbol c, then $t[a_1, \ldots, a_q] = I(c)$;

3. if t is $F^m(t_1, \ldots, t_m)$, where $t_1(x_1, \ldots, x_q), \ldots, t_m(x_1, \ldots, x_q)$ are terms, then

$$t[a_1, \ldots, a_q] = I(F^m)(\langle t_1[a_1, \ldots, a_q], \ldots, t_m[a_1, \ldots, a_q]\rangle).$$

2.

1. Suppose that $\varphi(x_1, \ldots, x_q)$ is an elementary formula $t_1 = t_2$, where $t_1(x_1, \ldots, x_q)$ and $t_2(x_1, \ldots, x_q)$ are terms. Then $a_1, \ldots, a_q$ satisfies φ if and only if

$$t_1[a_1, \ldots, a_q] = t_2[a_1, \ldots, a_q].$$

2. Suppose that $\varphi(x_1, \ldots, x_q)$ is an elementary formula $Q^n(t_1, \ldots, t_n)$, where Q^n is an n-place relation symbol and $t_1(x_1, \ldots, x_q), \ldots, t_n(x_1, \ldots, x_q)$ are terms. Then $a_1, \ldots, a_q$ satisfies φ if and only if

$$\langle t_1[a_1, \ldots, a_q], \ldots, t_n[a_1, \ldots, a_q]\rangle \in I(Q^n).$$

For brevity, we write

$$\mathcal{U} \vDash \varphi[a_1, \ldots, a_q]$$

for: $a_1, \ldots, a_q$ satisfies φ in $\mathcal{U}$.

3. Now suppose that φ is any formula of $\mathcal{L}$ and all its free and bound variables are among $x_1, \ldots, x_q$.

1. If φ is $\theta_1 \wedge \theta_2$, then

$$\mathcal{U} \vDash \varphi[a_1, \ldots, a_q] \text{ if and only if } \mathcal{U} \vDash \Theta_1[a_1, \ldots, a_q] \text{ and } \mathcal{U} \vDash \Theta_2[a_1, \ldots, a_q].$$

2. If φ is $\neg\Theta$, then

$$\mathcal{U} \vDash \varphi[a_1, \ldots, a_q] \text{ if and only if it is not true that } \mathcal{U} \vDash \Theta[a_1, \ldots, a_q].$$

3. If φ is $\forall x_i\, \psi$, where $i \leq q$, then

$$\mathcal{U} \vDash \varphi[a_1, \ldots, a_q] \text{ if and only if } \mathcal{U} \vDash \psi[a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q] \text{ for any } a \in A.$$

It is easy to check that the abbreviations $\vee$, $\Rightarrow$, $\Leftrightarrow$, and $\exists$ have their usual meanings. In particular, if φ is $\exists x_i\, \psi$, where $i \leq q$, then $\mathcal{U} \vDash \varphi[a_1, \ldots, a_q]$ if and only if there exists $a \in A$ such that

$$\mathcal{U} \vDash \psi[a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q].$$

The following proposition shows that the relation

$$\mathcal{U} \models \varphi(x_1, \ldots, x_p)[a_1, \ldots, a_q]$$

depends only on $a_1, \ldots, a_p$, where $p < q$.

Proposition 2. *1. Let $t(x_1, \ldots, x_p)$ be a term, and let $a_1, \ldots, a_q$ and $b_1, \ldots, b_r$ be two sequences of elements such that $p \leq q$, $p \leq r$, and $a_i = b_i$ whenever x_i is a free variable of the term t. Then*

$$t[a_1, \ldots, a_q] = t[b_0, \ldots, b_r].$$

2. Let φ be a formula, let all its variables, free and bound, belong to the set $x_1, \ldots, x_p$, and let $a_1, \ldots, a_q$ and $b_1, \ldots, b_r$ be two sequences of elements such that $p \leq q$, $p \leq r$, and $a_i = b_i$ whenever x_i is a free variable in the formula φ. Then

$$\mathcal{U} \models \varphi[a_1, \ldots, a_q] \quad \textit{if and only if} \quad \mathcal{U} \models \varphi[b_1, \ldots, b_r].$$

This proposition allows us to give the following definition. Let $\varphi(x_1, \ldots, x_p)$ be a formula, and let all its variables, free and bound, be contained in the set $x_1, \ldots, x_q$, where $p \leq q$. Let $a_1, \ldots, a_p$ be a sequence of elements of the set A. We shall say that φ *is true in* $\mathcal{U}$ *on* $a_1, \ldots, a_p$,

$$\mathcal{U} \models \varphi[a_1, \ldots, a_p]$$

if φ is true in $\mathcal{U}$ on $a_1, \ldots, a_p, \ldots, a_q$ with some (or, equivalently, any) sequence $a_{p+1}, \ldots, a_q$.

Let φ be a sentence, and let all its bound variables be contained in the set $x_1, \ldots, x_q$. We shall say that φ *is true in the model* $\mathcal{U}$ (notation: $\mathcal{U} \models \varphi$) if φ is true in $\mathcal{U}$ on some (equivalently, any) sequence $a_1, \ldots, a_q$.

This last phrase is equivalent to each of the following phrases:

φ holds in $\mathcal{U}$;
$\mathcal{U}$ satisfies φ;
$\mathcal{U}$ is a model of φ.

In the case where σ is not true in $\mathcal{U}$, we say that σ is *false* in $\mathcal{U}$, or that σ *does not hold* in $\mathcal{U}$, or that $\mathcal{U}$ *is a model of the sentence* $\neg\sigma$. If we have a set Σ of sentences, we say that $\mathcal{U}$ is a model of this set if $\mathcal{U}$ is a model of every sentence $\sigma \in \Sigma$. It is useful to denote this concept by $\mathcal{U} \models \Sigma$.

As we have said above, a *theory* T of the language $\mathcal{L}$ is a collection of sentences of the language $\mathcal{L}$. A *theory* of a model $\mathcal{U}$ (of the language $\mathcal{L}$) is the set of all sentences which hold in $\mathcal{U}$.

Two models $\mathcal{U}$ and $\mathcal{V}$ for $\mathcal{L}$ are called *elementarily equivalent* if every sentence that is true in $\mathcal{U}$ is true in $\mathcal{V}$, and vice versa. We express this relationship between models by $\equiv$. It is easy to see that $\equiv$ is indeed an equivalence relation. We can note that two models are elementarily equivalent if and only if their theories coincide.

Any two isomorphic models of the same language are elementarily equivalent. If two models of the same language are elementarily equivalent and one of them is finite, then these models are isomorphic. If models are infinite and elementarily equivalent, they are not necessarily isomorphic. For example, the field $\mathbb{C}$ of all complex numbers and the field $\bar{\mathbb{Q}}$ of all algebraic numbers are elementarily equivalent, but not isomorphic.

Together with first order languages we need to consider second order languages, in which we can also quantify relation symbols, i.e., use relation symbols as variables. Such languages will be described in Sec. 1.4.

1.4 Second Order Languages and Models

Now we shall introduce all notions, similar to the notions of Secs. 1.1 and 1.3, for second order languages and models.

A *second order language* $\mathcal{L}_2$ is a collection of symbols, consisting of (1) parentheses (,); (2) connectives $\wedge$ ("and") and $\neg$ ("not"); (3) the quantifier $\forall$ (for all); (4) the binary relation symbol $=$ (identity); (5) a countable set of object variables x_i; (6) a countable set of predicate variables P_i^l; (7) a finite or countable set of relation symbols Q_i^n ($n \geq 1$); (8) a finite or countable set of function symbols F_i^n ($n \geq 1$); (9) a finite or countable set of constant symbols c_i.

Terms of the language $\mathcal{L}_2$ are defined as follows:

1. a variable is a term;
2. a constant symbol is a term;
3. if F^n is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then $F^n(t_1, \ldots, t_n)$ is a term;
4. a symbol-string is a term only if it can be shown to be a term by a finite number of applications of (1)–(3).

Therefore terms of the language $\mathcal{L}_2$ coincide with terms of the language $\mathcal{L}$.

Elementary formulas of the language $\mathcal{L}_2$ are symbol-strings of the form given below:

1. if t_1 and t_2 are terms of the language $\mathcal{L}_2$, then $t_1 = t_2$ is an elementary formula;
2. if P^l is a predicate variable and $t_1, \ldots, t_l$ are terms, then the symbol-string $P^l(t_1, \ldots, t_l)$ is an elementary formula;
3. if Q^n is an n-place relation symbol, and $t_1, \ldots, t_n$ are terms, then the symbol-string $Q^n(t_1, \ldots, t_n)$ is an elementary formula.

Therefore elementary formulas of the second order group language have the form $x_i = x_j$, $x_i = x_j \cdot x_k$, and $P^l(x_{i_1}, \ldots, x_{i_l})$, where $l \geq 1$.

Formulas of the language $\mathcal{L}_2$ are defined as follows:

1. an elementary formula is a formula;
2. if φ and ψ are formulas and x is an object variable, then $(\neg\varphi)$, $(\varphi \wedge \psi)$, and $(\forall x\, \varphi)$ are formulas;
3. if P^l is a predicate variable and φ is a formula, then the symbol-string $(\forall P^l(v_1, \ldots, v_l)\varphi)$ is a formula;

4. a symbol-string is a formula only if it can be shown to be a formula by a finite number of applications of (1)–(3).

Let us introduce the following abbreviations:

$\exists P^l(v_1, \ldots, v_l)\, \varphi$ is an abbreviation for $\neg(\forall P^l(v_1, \ldots, v_l)\,(\neg\varphi))$;

$\forall P_1^{l_1}(v_1, \ldots, v_{l_1}) \ldots \forall P_n^{l_n}(v_1, \ldots, v_{l_n})\, \varphi$ is an abbreviation for

$$(\forall P_1^{l_1}(v_1, \ldots, v_{l_1})) \ldots (\forall P_n^{l_n}(v_1, \ldots, v_{l_n}))\, \varphi;$$

$\exists P_1^{l_1}(v_1, \ldots, v_{l_1}) \ldots \exists P_n^{l_n}(v_1, \ldots, v_{l_n})\, \varphi$ is an abbreviation for

$$(\exists P_1^{l_1}(v_1, \ldots, v_{l_1})) \ldots (\exists P_n^{l_n}(v_1, \ldots, v_{l_n}))\, \varphi.$$

Introduce the notions of *free* and *bound* occurrence of a predicate variable in a formula of the language $\mathcal{L}_2$.

1. All occurrences of all predicate variables in elementary formulas are free occurrences.
2. Every free (bound) occurrence of a variable P^l in a formula φ is a free (bound) occurrence of a variable P^l in the formulas $(\neg\varphi)$, $(\varphi \wedge \psi)$, and $(\psi \wedge \varphi)$.
3. For any occurrence of a variable P^l in a formula φ, the occurrence of the variable P^l in the formula $\forall P^l(v_1, \ldots, v_l)\, \varphi$ is bound. If an occurrence of a variable P_1^l in a formula φ is free (bound), then the occurrences of P_1^l in the formulas $\forall x\, \varphi$ and $\forall P_2^m(v_1, \ldots, v_m)\, \varphi$ are free (bound).

As in Sec. 1.1, any formula such that all its *free* object and predicate variables are among the set $\{x_1, \ldots, x_n, P_1^{l_1}, \ldots, P_k^{l_k}\}$ will be denoted by $\varphi(x_1, \ldots, x_n, P_1^{l_1}, \ldots, P_k^{l_k})$.

To our five purely logical axioms from Sec. 1.1 we shall add the sixth purely logical axiom:

6. $(\forall P^n(v_1, \ldots, v_n)\,(\psi \Rightarrow \varphi) \Rightarrow (\psi \Rightarrow (\forall P^n(v_1, \ldots, v_n)\, \varphi))$ if ψ does not contain any free occurrences of the variable P^n.

To the identity axioms we add the fourth identity axiom:

4. $\forall P^n(v_1, \ldots, v_n)\,(y = z \Rightarrow (P^n(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \Leftrightarrow P^n(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n))$.

The rule of generalization can be changed to "from φ infer $\forall x\, \varphi$ and $\forall P^n(v_1, \ldots, v_n)\, \varphi$."

A *model of a second order language* $\mathcal{L}_2$ (see Sec. 1.3) is a pair $\mathcal{U} = \langle A, I\rangle$ consisting of an object A (i.e., some class or set of the theory NBG) and some correspondence I that assigns to every relation symbol Q^n some n-place relation in A, to every function symbol F^n some n-place function in A, and to every constant symbol c some element of A.

Now we shall give a definition of satisfaction. Let φ be any formula of the language $\mathcal{L}_2$ such that all its free and bound variables are among $x_1, \ldots, x_q, P_1^{l_1}, \ldots, P_s^{l_s}$, and let $a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s}$ be any sequence, where $a_1, \ldots, a_q$ are elements of the set A, $b_i^{l_i} \subset A^{l_i}$. We define the predicate

φ is satisfied by the sequence $a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s}$ in the model $\mathcal{U}$.

1. The value of a term $t(x_1, \ldots, x_q, P_1^{l_1}, \ldots, P_s^{l_s})$ at $a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s}$ is defined as follows (we let $t[a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s}]$):

1. if t is a variable x_i, then $t[a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s}] = a_i$;
2. if t is a constant symbol c, then $t[a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s}] = I(c)$;

3. if t is $F^m(t_1, \dots, t_m)$, where $t_1(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s}), \dots, t_m(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ are terms, then $t[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] = I(F^m)(\langle t_1[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}], \dots, t_m[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}]\rangle)$.

2.

1. Suppose that $\varphi(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ is an elementary formula $t_1 = t_2$, where $t_1(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ and $t_2(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ are terms. Then $a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}$ satisfies φ if and only if
$$t_1[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] = t_2[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}].$$

2. Suppose that $\varphi(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ is an elementary formula $Q^n(t_1, \dots, t_n)$, where Q^n is an n-place relation symbol and $t_1(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s}), \dots, t_n(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ are terms. Then $a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}$ satisfies φ if and only if
$$\langle t_1[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}], \dots, t_n[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}]\rangle \in I(Q^n).$$

3. Suppose that $\varphi(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ is an elementary formula $P_i^{l_i}(t_1, \dots, t_{l_i})$, where $t_1(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s}), \dots, t_n(x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s})$ are terms. Then $a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}$ satisfies φ if and only if
$$\langle t_1[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}], \dots, t_n[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}]\rangle \in b_i^{l_i}.$$

3. Now suppose that φ is any formula such that all its free and bound variables are among $x_1, \dots, x_q, P_1^{l_1}, \dots, P_s^{l_s}$.

1. If φ is $\theta_1 \wedge \theta_2$, then
$$\mathcal{U} \vDash \varphi[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] \text{ if and only if } \mathcal{U} \vDash \Theta_1[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] \text{ and } \mathcal{U} \vDash \Theta_2[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}].$$

2. If φ is $\neg\Theta$, then
$$\mathcal{U} \vDash \varphi[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] \text{ if and only if it is false that } \mathcal{U} \vDash \Theta[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}].$$

3. If φ is $\forall x_i \psi$, where $i \le q$, then
$$\mathcal{U} \vDash \varphi[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] \text{ if and only if } \mathcal{U} \vDash \psi[a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] \text{ for any } a \in A.$$

4. If φ is $\forall P_i^{l_i}(v_1, \dots v_{l_i})\psi$, where $i \le s$, then
$$\mathcal{U} \vDash \varphi[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}] \text{ if and only if } \mathcal{U} \vDash \psi[a_1, \dots, a_q, b_1^{l_1}, \dots, b_{i-1}^{l_{i-1}}, b, b_{i+1}^{l_{i+1}}, \dots, b_s^{l_s}] \text{ for any } b \subset A^{l_i}.$$

The proposition that
$$\mathcal{U} \models \varphi(x_1, \dots, x_p, P_1^{l_1}, \dots, P_t^{l_t})[a_1, \dots, a_q, b_1^{l_1}, \dots, b_s^{l_s}]$$
depends only on the values $a_1, \dots, a_p, b_1^{l_1}, \dots, b_t^{l_t}$, where $p < q$, $s < t$, is the same as Proposition 2.

All other definitions are also similar to definitions from Sec. 1.3.

We shall say that two models of the second order language $\mathcal{L}_2$ are equivalent in $\mathcal{L}_2$ if for every sentence of this language the sentence is true in one model if and only if it is true in the other model.

2 Basic Concepts about Abelian Groups

2.1 Preliminaries

The word “group” will mean, throughout, an additively written Abelian (i.e., commutative) group. That is, by group we shall understand a set A such that with every pair of elements $a, b \in A$ there is associated an element $a+b$ of A, which is called the *sum* of elements a and b; there is an element $0 \in A$, the *zero*, such that $a+0=a$ for every $a \in A$; for each $a \in A$ there is an $x \in A$ with the property $a+x=0$, this $x=-a$ is called the *inverse* (*opposite*) to a; finally, we have both commutative and associative laws: $a+b=b+a$, $(a+b)+c=a+(b+c)$ for every $a, b, c \in A$.

A sum $a+\cdots+a$ (n times) is abbreviated as na, and $-a-\cdots-a$ (n times) as $-na$. By the *order* of a group A we mean the power $|A|$ of the set of its elements. If the power $|A|$ is a finite (countable) cardinal, then the group A is called *finite* (*countable*).

A subset B of A is a *subgroup* if $\forall b_1, b_2 \in B\,(b_1+b_2 \in B)$. If B is a subgroup consisting of the zero alone or of all elements of A, then B is a *trivial* subgroup of A; but a subgroup of A that is different from A is called a *proper* subgroup of A. We shall write $B \lhd A$ to indicate that B is a subgroup of A. Let $B \lhd A$ and $a \in A$. The set $a+B=\{a+b \mid b \in B\}$ is said to be a *coset of A modulo B*. An element of the coset is called a *representative* of this coset. A set consisting of representatives, one for each coset of A modulo B, is called the *complete system of representatives of cosets modulo B*. Its power is called the index of B in A, and denoted by $|A:B|$.

The cosets of A modulo B form a group A/B known as the *quotient group* (of A with respect to B). If S is any subset in A, then by $\langle S\rangle$ we denote the subgroup of A, *generated* by S, i.e., the intersection of all subgroups of A containing S. In particular, if S consists of the elements a_i ($i \in I$), we also write $\langle S\rangle=\langle \ldots, a_i, \ldots\rangle_{i\in I}$ or simply $\langle S\rangle=\langle a_i\rangle_{i\in I}$. The subgroup $\langle S\rangle$ consists exactly of all finite linear combinations of the elements of S, i.e., of all sums $n_1a_1+\cdots+n_ka_k$ with $a_i \in S$, n_i integers, and k an arbitrary nonnegative integer. If S is empty, then we put $\langle S\rangle=0$. In the case $\langle S\rangle=A$ we say S is a *generating system* of A and the elements of S are *generators* of A. If there is a finite generating system, then A is said to be a *finitely generated* group.

If B and C are two subgroups of A, then the subgroup $\langle B, C\rangle$ generated by them consists of all elements of A of the form $b+c$, where $b \in B$ and $c \in C$. Therefore we shall write $\langle B, C\rangle=B+C$. Similarly, for some set of subgroups B_i of A we shall write $B=\langle B_i\rangle_{i\in I}=\sum\limits_{i\in I} B_i$.

The group $\langle a\rangle$ is the *cyclic* group generated by a. The order of $\langle a\rangle$ is also called the *order* of a (notation: $o(a)$). If every element of A is of finite order, then A is called a *torsion* or *periodic* group. If all elements of A, except 0, are of infinite order. then A is *torsion free. Mixed groups* contain both nonzero elements of finite order and elements of infinite order.

A *primary group* or *p-group* is defined to be a group, the orders of whose elements are powers of a fixed prime p.

Given $a \in A$, the greatest nonnegative integer r for which $p^r x=a$ has a solution $x \in A$ is called the *p-height* $h_p(a)$ of a. If $p^r x=a$ is solvable whatever r is, then a is of *infinite p-height*, $h_p(a)=\infty$. If it is completely clear from the context which prime p is meant, then we call $h_p(a)$ simply the *height* of a and write $h(a)$. For a group A and an integer $n>0$, let $nA=\{na \mid a \in A\}$ and $A[n]=\{a \mid a \in A,\ na=0\}$.

A map $\alpha\colon A \to B$ is called a *homomorphism* (of A into B) if

$$\forall a_1, a_2 \in A \quad \alpha(a_1+a_2)=\alpha a_1+\alpha a_2.$$

The *kernel* of α ($\operatorname{Ker}\alpha \lhd A$) is the set of all elements $a \in A$ such that $\alpha a=0$. The *image* of α ($\operatorname{Im}\alpha \lhd B$) consists of all $b \in B$ that for some $a \in A$ satisfy $\alpha a=b$. If $\operatorname{Im}\alpha=B$, then α is called a *surjective* homomorphism, or an *epimorphism.* If $\operatorname{Ker}\alpha=0$, then α is called an *injective* homomorphism, or a *monomorphism.* A homomorphism that is injective and surjective simultaneously is called an *isomorphism.*

Now we consider the most important types of Abelian groups.

Cyclic groups were defined above as groups $\langle a\rangle$ for some a. Note that all subgroups of cyclic groups are likewise cyclic.

For a fixed prime p, consider the p^nth complex roots of unity, $n \in \mathbb{N}\cup\{0\}$. They form an infinite multiplicative group; in accordance with our convention, we switch to the additive notation. This group, called

a *quasicyclic* group or a *group of type* p^∞ (notation: $\mathbb{Z}(p^\infty)$) can be defined as follows. It is generated by elements $c_1, c_2, \dots, c_n, \dots$, such that $pc_1 = 0$, $pc_2 = c_1, \dots, pc_{n+1} = c_n, \dots$. Here $o(c_n) = p^n$, and every element of $\mathbb{Z}(p^\infty)$ is a multiple of some c_n. It is clear that all the quasicyclic groups corresponding to the same prime p are isomorphic.

Let p be a prime, and Q_p be the ring of rational numbers whose denominators are prime to p. The nonzero ideals of Q_p are principal ideals generated by p^k with $k = 0, 1, \dots$. If one considers the ideals (p^k) as a fundamental system of neighborhoods of 0, then Q_p becomes a topological ring, and we may form the completion Q_p^* of Q_p in this topology. Q_p^* is again a ring whose ideals are (p^k) with $k = 0, 1, \dots$, and which is complete in the topology defined by its ideals.

The elements of Q_p^* may be represented as follows: let $\{t_0, t_1, \dots, t_{p-1}\}$ be a complete set of representatives of cosets of $p^k Q_p$ modulo $p^{k+1} Q_p$. Let $\pi \in Q_p^*$, and let $a_n \in Q_p$ be a sequence tending to π. According to the definition of fundamental sequence, almost all its elements (i.e., all with a finite number of exceptions) belong to the same coset modulo pQ_p, say, to that represented by s_0. Almost all differences $a_n - s_0$ belonging to pQ_p belong to the same coset of the ring pQ_p modulo $p^2 Q_p$, say, to that represented by ps_1. So proceeding, π uniquely defines a sequence $s_0, ps_1, \dots$, and we associate with π the formal infinite series $s_0 + s_1 p + \dots$. Its partial sums $b_n = s_0 + s_1 p + \dots + s_n p^n$ form in Q_p a fundamental sequence which converges in Q_p^* to π, in view of $\pi - b_n \in p^k Q_p^*$. From the uniqueness of limits it follows that, in this way, different elements of Q_p^* are associated with different series. Therefore we can identify the elements π of the ring Q_p^* with the formal series $s_0 + s_1 p + s_2 p^2 + \dots$ with coefficients from $\{0, 1, \dots, p-1\}$ and write

$$\pi = s_0 + s_1 p + s_2 p^2 + \dots \quad (s_n = 0, 1, \dots, p-1).$$

The arising ring is a *commutative integral domain* (i.e., the commutative ring without zero divisors) and is called the *ring of p-adic integers*.

2.2 Direct Sums

Let B and C be two subgroups of A such that:

1. $B + C = A$;

2. $B \cap C = 0$.

Then we call A the *direct sum* of its subgroups B and C ($A = B \oplus C$).

If the condition (2) is satisfied, then we say that the groups B and C *are disjoint*.

If B_i $(i \in I)$ is a family of subgroups of A such that

1. $\sum\limits_{i \in I} B_i = A$;

2. $\forall i \in I \ B_i \cap \sum\limits_{j \neq i} B_j = 0$,

then we say that the group A is a *direct sum* of its subgroups B_i, and write $A = \bigoplus\limits_{i \in I} B_i$, or $A = B_1 \oplus \dots \oplus B_n$, if $I = \{1, \dots, n\}$. A subgroup B of the group A is called a *direct summand* of A if there exists a subgroup $C \lhd A$ such that $A = B \oplus C$. In this case, C is called the *complementary direct summand* or simply the *complementation*.

Two direct decompositions of A, $A = \bigoplus\limits_i B_i$ and $A = \bigoplus\limits_j C_j$, are called *isomorphic* if the components B_i and C_j may be brought into a one-to-one correspondence such that the corresponding components are isomorphic.

If we have two groups B and C, then the set of all pairs (b, c), where $b \in B$ and $c \in C$, forms a group A if we set $(b_1, c_1) = (b_2, c_2)$ if and only if $b_1 = b_2$ and $c_1 = c_2$, and $(b_1, c_1) + (b_2, c_2) = (b_1 + b_2, c_1 + c_2)$.

The correspondences $b \mapsto (b, 0)$ and $c \mapsto (0, c)$ are isomorphisms between the groups B, C and the subgroups B', C' of A. We can write $A = B \oplus C$ and call A the (*external*) *direct sum* of B and C.

Let B_i $(i \in I)$ be a set of groups. A *vector* $(\dots, b, \dots)$ over the groups B_i is a vector with ith coordinate for every $i \in I$ equal to some $b_i \in B_i$. Equality and addition of vectors is defined coordinatewise. In this way, the set of all vectors becomes a group C, called the *direct product* of the groups B_i,

$$C = \prod_{i \in I} B_i.$$

The correspondence $\rho_i \colon b_i \mapsto (\dots, 0, b_i, 0, \dots)$, where b_i stands on the ith place, and 0 everywhere else, is an isomorphism of the group B_i with a subgroup B'_i of C. The groups B'_i $(i \in I)$ generate in C the group A of all vectors $(\dots, b_i, \dots)$ with $b_i = 0$ for almost all $i \in I$, and $A = \bigoplus_{i \in I} B'_i$. The group A is also called the (*external*) *direct sum* of B_i. If A is a group and $\varkappa$ is a cardinal number, then by $\bigoplus_{\varkappa} A$ we shall denote the direct sum of $\varkappa$ groups isomorphic to A, and by $\prod_{\varkappa} A = A^{\varkappa}$ we shall denote the direct product of $\varkappa$ such groups.

Proposition 1 ([15]). *Every torsion group A is a direct sum of p-groups A_p belonging to different primes p. The groups A_p are uniquely determined by A.*

The subgroups A_p are called the *p-components* of A. By virtue of Proposition 1, the theory of torsion groups is essentially reduced to that of primary groups.

Proposition 2 ([15]). *If there exists a projection π of the group A onto its subgroup B, then B is a direct summand in A.*

2.3 Direct Sums of Cyclic Groups

Proposition 3 ([15]). *For a group A the following conditions are equivalent*:

1. *A is a finitely generated group*;
2. *A is a direct sum of a finite number of cyclic groups.*

A system $\{a_1, \dots, a_k\}$ of nonzero elements of a group A is called *linearly independent* if

$$n_1 a_1 + \dots + n_k a_k = 0 \quad (n_i \in \mathbb{Z})$$

implies

$$n_1 a_1 = \dots = n_k a_k = 0.$$

A system of elements is *dependent* if it is not independent.

An infinite system $L = \{a_i\}_{i \in I}$ of elements of the group A is called *independent* if every finite subsystem of L is independent. An independent system M of A is *maximal* if there is no independent system in A containing M properly. By the *rank* $r(A)$ of a group A we mean the cardinality of a maximal independent system containing only elements of infinite and prime power orders.

Proposition 4. *The rank $r(A)$ of a group A is an invariant of this group.*

Theorem 1 ([23, 1]). *A bounded group is a direct sum of cyclic groups.*

Theorem 2 ([15]). *Any two decompositions of a group into direct sums of cyclic groups of infinite and prime power orders are isomorphic.*

Theorem 3 ([19]). *Subgroups of direct sums of cyclic groups are again direct sums of cyclic groups.*

2.4 Divisible Groups

We shall say that an element a of A is *divisible by* a positive integer n ($n \mid a$) if there is an element $x \in A$ such that $nx = a$. A group D is called *divisible* if $n \mid a$ for all $a \in D$ and all natural n. The groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are examples for divisible groups.

Theorem 4. *Any divisible group D is a direct sum of quasicyclic groups and full rational groups $\mathbb{Q}$. The powers of the sets of components $\mathbb{Z}(p^\infty)$ (for every p) and $\mathbb{Q}$ form a complete and independent system of invariants for the group D.*

Corollary. *Any divisible p-group D is a direct sum of the groups $\mathbb{Z}(p^\infty)$. The power of the set of components $\mathbb{Z}(p^\infty)$ is the only invariant of D.*

Theorem 5 ([15]). *For a group D the following conditions are equivalent*:

1. *D is a divisible group*;
2. *D is a direct summand in every group containing D.*

2.5 Pure Subgroups

A subgroup G of A is called *pure* if the equation $nx = g \in G$ is solvable in G whenever it is solvable in the whole group A. In other words, G is pure if and only if

$$\forall n \in \mathbb{Z} \quad nG = G \cap nA.$$

Proposition 5 ([26]). *Assume that a subgroup B of A is a direct sum of cyclic groups of the same order p^k. Then the following statements are equivalent*:

1. *B is a pure subgroup of A*;
2. *B is a direct summand of A.*

Corollary. *Every element of order p and of finite height can be embedded in a finite cyclic direct summand of the group.*

Theorem 6 ([18]). *A bounded pure subgroup is a direct summand.*

Corollary 1 ([12]). *A p-subgroup B of a group A can be embedded in a bounded direct summand of A if and only if the heights of the nonzero elements of B (relative to A) are bounded.*

Corollary 2. *An element a of prime power order belongs to a finite direct summand of A if and only if $\langle a \rangle$ contains no elements of infinite height.*

2.6 Basic Subgroups

A subgroup B of a group A is called a *p-basic subgroup* if it satisfies the following conditions:

1. B is a direct sum of cyclic p-groups and infinite cyclic groups;
2. B is pure in A;
3. A/B is p-divisible.

According to this definition, B possesses a basis which is said to be a *p-basis* of A.

Every group, for every prime p, contains p-basic subgroups [14].

We now focus our attention on p-groups, where p-basic subgroups are particularly important. If A is a p-group and q is a prime different from p, then evidently A has only one q-basic subgroup, namely 0. Therefore, in p-groups we may refer to the p-basic subgroups simply as *basic* subgroups, without danger of confusion.

Theorem 7 ([27]). *Assume that B is a subgroup of a p-group A, $B = \bigoplus_{n=1}^{\infty} B_n$, and B_n is a direct sum of groups $\mathbb{Z}(p^n)$. Then B is a basic subgroup of A if and only if for every integer $n > 0$ the subgroup $B_1 \oplus \cdots \oplus B_n$ is a maximal p^n-bounded direct summand of A.*

Theorem 8 (Bear, Boyer [4]). *Suppose that B is a subgroup of a p-group A,*

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \ldots,$$

where

$$B_n \cong \bigoplus_{\mu_n} \mathbb{Z}(p^n).$$

The subgroup B is a basic subgroup of A if and only if

$$A = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus (B_n^* + p^n A),$$

where $n \in \mathbb{N}$,

$$B_n^* = B_{n+1} \oplus B_{n+2} \oplus \ldots.$$

Since the group B has a basis and the factor group A/B is a direct sum of groups isomorphic to $\mathbb{Z}(p^\infty)$ (i.e., A/B also has a generating system which can be easily described), it is natural to combine these generating systems and to obtain one for A.

We write

$$B = \bigoplus_{i \in I} \langle a_i \rangle \quad \text{and} \quad A/B = \bigoplus_{j \in J} C_j^*, \quad \text{where} \quad C_j^* = \mathbb{Z}(p^\infty).$$

If the direct summand C_j^* is generated by cosets $c_{j1}^*, \ldots, c_{jn}^*, \ldots$ modulo B with $pc_{j1}^* = 0$ and $pc_{j,n+1}^* = c_{jn}^*$ $(n = 1, 2, \ldots)$, then, by the purity of B in A, in the group A we can pick out $c_{jn} \in c_{jn}^*$ of the same order as c_{jn}^*. Then we get the following set of relations:

$$pc_{j1} = 0, \quad pc_{j,n+1} = c_{jn} = b_{jn} \quad (n \geq 1, \ b_{jn} \in B),$$

where b_{jn} must be of order at most p^n, since $o(c_{jn}) = p^n$.

The system $\{a_i, c_{jn}\}_{i \in I,\ j \in J,\ n \in \omega}$ will be called a *quasibasis* of A.

Proposition 6 ([13]). *If $\{a_i, c_{jn}\}$ is a quasibasis of the p-group A, then every $a \in A$ can be written in the following form:*

$$a = s_1 a_{i_1} + \cdots + s_m a_{i_m} + t_1 a_{j_1 n_1} + \cdots + t_r a_{j_r n_r}, \tag{1}$$

where s_i and t_j are integers, no t_j is divisible by p, and the indices $i_1, \ldots, i_m$ as well as $j_1, \ldots, j_r$ are distinct. The expression (1) *is unique in the sense that a uniquely defines the terms sa_i and tc_{jn}.*

Theorem 9 (Kulikov [20]). *If B is a basic subgroup of a reduced p-group A, then*

$$|A| \leq |B|^\omega.$$

The *final rank* of a basic subgroup B of a p-group A is the infimum of the cardinals $r(p^n B)$. Note that if the rank of B is equal to μ_1 and the final rank of B is equal to μ_2, then $A = A_1 \oplus A_2$, where the group A_1 is bounded and has the rank μ_1, and a basic subgroup of the group A_2 has the rank μ_2 which coincides with its final rank.

Theorem 10. *If two endomorphisms of a reduced Abelian group coincide on some of its basic subgroups, then they are equal.*

2.7 Endomorphism Rings of Abelian Groups

If we have an Abelian group A, then its endomorphisms form a ring with respect to the operations of addition and composition of homomorphisms. This ring will be denoted by $\mathrm{End}(A)$.

We need some facts about the ring $\mathrm{End}(A)$.

1. There exists a one-to-one correspondence between finite direct decompositions
$$A = A_1 \oplus \cdots \oplus A_n$$
of the group A and decompositions of the ring $\mathrm{End}(A)$ in finite direct sums of left ideals
$$\mathrm{End}(A) = L_1 \oplus \cdots \oplus L_n;$$
namely, if $A_i = e_i A$, where $e_1, \ldots, e_n$ are mutually orthogonal idempotents, then $L_i = \mathrm{End}(A)e_i$.

2. An idempotent $e \neq 0$ is called primitive if it can not be represented as a sum of two nonzero orthogonal idempotents. If $e \neq 0$ is an idempotent of the ring $\mathrm{End}(A)$, then eA is an indecomposable direct summand of A if and only if e is a primitive idempotent.

3. Let $A = B \oplus C$ and $A = B' \oplus C'$ be direct decompositions of the group A, and let $e\colon A \to B$ and $e'\colon A \to B'$ be the corresponding projections. Then $B \cong B'$ if and only if there exist elements $\alpha, \beta \in \mathrm{End}(A)$ such that
$$\alpha\beta = e \text{ and } \beta\alpha = e'.$$

Theorem 11 (Baer [2], Kaplansky [17]). *If A and C are torsion groups, and* $\mathrm{End}(A) \cong \mathrm{End}(C)$*, then* $A \cong C$.

Theorem 12 (Charles [11], Kaplansky [17]). *The center of the endomorphism ring* $\mathrm{End}(A)$ *of a p-group A is the ring of p-adic integers or the residue class ring of the integers modulo p^k, depending on whether A is unbounded or bounded with p^k as the least upper bound for the orders of its elements.*

3 Beautiful Linear Combinations

In this section, we shall completely follow the paper [24] of S. Shelah.

Suppose that we have some fixed Abelian p-group $A \cong \bigoplus_\mu \mathbb{Z}(p^l)$, where μ is an infinite cardinal number and $\mathrm{End}(A)$ is its endomorphism ring.

For any $f \in \mathrm{End}(A)$ let $\mathrm{Rng}\, f$ be its range in A and $\mathrm{Cl}\, B$ (or $\langle B \rangle$) be the closure of $B \subset A$ in A, i.e., the minimal subgroup in A containing B.

As usual, $\vec{x}$ denotes a finite sequence of variables $\vec{x} = \langle x_1, \ldots, x_n \rangle$. A linear combination $k_1 x_1 + \cdots + k_n x_n$, where $k_i \in \mathbb{Z}$, will also be denoted by $\tau(x_1, \ldots, x_n)$, or $\tau(\vec{x})$. Such a combination will be called *reduced* if all k_i are distinct and not equal to zero.

Let $\{a_i \mid i \in I\}$ be some independent subset of elements of order p^l in the group A. It is clear that every function $h\colon \{a_i \mid i \in I\} \to A$ has a unique extension
$$\tilde{h} \in \mathrm{Hom}(\mathrm{Cl}\{a_i \mid i \in I\}, A).$$

Let B be some set and h be a function from B into B. For every $x \in B$, we define its *depth* ($\mathrm{Dp}(x) = \mathrm{Dp}(x, h)$) as the least ordinal number (or infinity) satisfying the following conditions:

1. $\mathrm{Dp}(x) \geq 0$ if and only if $x \in B$;

2. $\mathrm{Dp}(x) \geq \delta$ if and only if $\mathrm{Dp}(x) \geq \alpha$ for every $\alpha \in \delta$ such that δ is a limit ordinal number;

3. $\mathrm{Dp}(x) \geq \alpha + 1$ if and only if for some $y \in B$ we have $h(y) = x$ and $\mathrm{Dp}(y) \geq \alpha$.

Lemma 1. *Let $\{a_i \mid i \in I\} \subset A$ be an independent set consisting of elements of order p^l, and h be a function from I into I. Define $\tilde{h}$ by the formula*

$$\tilde{h}(k_1 a_{i_1} + \cdots + k_n a_{t_n}) = k_1 a_{h(t_1)} + \cdots + k_n a_{h(t_n)}.$$

Then

a. $\tilde{h} \in \mathrm{End}(\mathrm{Cl}\{a_i \mid i \in I\})$;

b. $\mathrm{Dp}(k_1 a_{t_1} + \cdots + k_n a_{t_n}, \tilde{h}) \geq \min_{i \in \{1,\dots,n\}} \mathrm{Dp}(t_i, h)$;

c. if in (b) *the linear combination $k_1 a_{t_1} + \cdots + k_n a_{t_n}$ is reduced and t_i are distinct, then the equality holds.*

Proof. (a) Immediate.

(b) We prove by induction on ordinal numbers α that

$$\min_{i \in \{1,\dots,n\}} \mathrm{Dp}(t_i, h) \geq \alpha \Rightarrow \mathrm{Dp}(k_1 a_{t_1} + \cdots + k_n a_{t_n}, \tilde{h}) \geq \alpha.$$

This suffices for the proof of (b). If $\alpha = 0$ or α is a limit ordinal, then this is trivial. For $\alpha = \beta + 1$, by the assumption and definition of depth, there are $s_i \in I$ such that $h(s_i) = t_i$ and $\mathrm{Dp}(s_i, h) \geq \beta$. Then $\min\limits_i \mathrm{Dp}(s_i, h) \geq \beta$, whence by the induction hypothesis

$$\mathrm{Dp}(k_1 a_{s_1} + \cdots + k_n a_{s_n}, \tilde{h}) \geq \beta,$$

but

$$\tilde{h}(k_1 a_{s_1} + \cdots + k_n a_{s_n}) = k_1 a_{t_1} + \cdots + k_n a_{t_n},$$

whence

$$\mathrm{Dp}(k_1 a_{t_1} + \cdots + k_n a_{t_n}, \tilde{h}) \geq \beta + 1 = \alpha.$$

(c) It suffices to prove by induction on α that

$$\mathrm{Dp}(k_1 a_{t_1} + \cdots + k_n a_{t_n}, \tilde{h}) \geq \alpha \Rightarrow \mathrm{Dp}(t_i, h) \geq \alpha$$

for all i. If $\alpha = 0$ or α is a limit ordinal, then this is trivial. For $\alpha = \beta + 1$, by the definition of depth, there are a reduced linear combination $l_1 a_{s_1} + \cdots + l_m a_{s_m}$ and distinct s_i such that

1. $\tilde{h}(l_1 a_{s_1} + \cdots + l_m a_{s_m}) = k_1 a_{t_1} + \cdots + k_n a_{t_n}$ and
2. $\mathrm{Dp}(l_1 a_{s_1} + \cdots + l_m a_{s_m}, \tilde{h}) \geq \beta$.

By (2) and the induction hypothesis, for $i = 1, \dots, m$ we have $\mathrm{Dp}(s_i, h) \geq \beta$.
By (1) and the definition of $\tilde{h}$,

$$l_1 a_{h(s_1)} + \cdots + l_m a_{h(s_m)} = k_1 a_{t_1} + \cdots + k_n a_{t_n}.$$

As the linear combination $k_1 a_{t_1} + \cdots + k_n a_{t_n}$ is reduced and the t_i are distinct,

$$\{t_1, \dots, t_n\} \subseteq \{h(s_1), \dots, h(s_m)\}.$$

Thus, for each $i = 1, \dots, n$ there is k_i, $1 \leq k_i \leq m$, such that $t_i = h(s_{k_i})$. Hence

$$\mathrm{Dp}(t_i, h) \geq \mathrm{Dp}(s_{k_i}, h) + 1 \geq \beta + 1 = \alpha. \quad \square$$

Lemma 2. *Let h_1 and h_2 be commuting functions from B into B. Then for any $x \in B$ we have*

$$\mathrm{Dp}(x, h_1) \leq \mathrm{Dp}(h_2(x), h_1).$$

Proof. We prove by induction on α that

$$\mathrm{Dp}(x, h_1) \geq \alpha \Rightarrow \mathrm{Dp}(h_2(x), h_1) \geq \alpha.$$

If $\alpha = 0$ or α is a limit ordinal, this is immediate.

Now we consider $\alpha = \beta + 1$.

If $\mathrm{Dp}(x, h_1) \geq \beta + 1$, then for some $y \in B$ we have $h_1(y) = x$ and $\mathrm{Dp}(y, h_1) \geq \beta$. Thus $h_1(h_2(y)) = h_1 \circ h_2(y) = h_2 \circ h_1(y) = h_2(x)$, and by the induction hypothesis $\mathrm{Dp}(h_2(y), h_1) \geq \beta$ (since $\mathrm{Dp}(y, h_1) \geq \beta$)), whence $\mathrm{Dp}(h_2(x), h_1) \geq \mathrm{Dp}(h_2(y), h_1) + 1 \geq \beta + 1 = \alpha$. $\square$

Lemma 3. *Let $\{a_i \mid i \in I\} \subset A$ be an independent set with elements of the same order p^l and let $A' = \mathrm{Cl}\{a_i \mid i \in I\}$. Let $J \subseteq I$, $|I \setminus J| = |I|$, $J = \bigcup_{\alpha \in \alpha(0)} J_\alpha$, and $B = \mathrm{Cl}\{a_i \mid i \in J\}$. Then we can find $f \in \mathrm{End}(A')$ such that*

a. $i \in J_\alpha \Rightarrow \mathrm{Dp}(a_i, f) = \alpha$;

b. if $g \in \mathrm{End}(A')$, $g \circ f = f \circ g$, and g maps B into B, then for every $\alpha \in \alpha(0)$

$$i \in J_\alpha \Rightarrow g(a_i) \in \mathrm{Cl}\left\{a_j \;\middle|\; j \in \bigcup_{\alpha \leq \beta < \alpha(0)} J_\beta\right\};$$

c. every function $g\colon \{a_i \mid i \in J\} \to B$ satisfying the condition (b) *can be extended to an endomorphism of $\mathrm{End}(A')$ that commutes with f and maps B into B.*

Proof. By renaming we can assume that $I \setminus J = I_0 \cup \{\langle 0, t, \eta\rangle \mid t \in J_\alpha,\ \alpha < \alpha(0),\ l(\eta) > 0,\ \eta$ is a decreasing sequence of ordinals, $\eta_0 < \alpha\} \cup \{\langle 1, t, n\rangle \mid 0 < n < \omega\}$ and we identify $\langle 0, t, \langle\ \rangle\rangle$ and $\langle 1, t, 0\rangle$ with t. Let us define a function h on I:

1. for $s \in I_0$
$$h(s) = s;$$

2. for $t \in J$, $\langle 0, t, \langle \eta, \beta\rangle\rangle \in I$, where β is an ordinal number,
$$h(\langle 0, t, \langle \eta, \beta\rangle\rangle) = \langle 0, t, \eta\rangle;$$

3. for $t \in J$
$$h(\langle 1, t, n\rangle) = \langle 1, t, n + 1\rangle.$$

Let $f = \tilde{h}$. It is easy to prove that if $t \in J$ and $l(\eta) = n$, then $\mathrm{Dp}(\langle 0, t, \eta\rangle, h) = \eta_n$. Hence (a) follows immediately by (c) of Lemma 1, and $f \in \mathrm{End}(A')$ by (a) of Lemma 1.

Let now $g \in \mathrm{End}(A')$ commute with f, g map B into B, $i \in J_\alpha$, $\alpha \in \alpha_0$. Suppose that the linear combination $g(a_i) = k_1 a_{s_1} + \cdots + k_n a_{s_n}$ is reduced. Since g maps B into B and $g(a_i) \in B$, we see that $s_k \in J$ for $k = 1, \ldots, n$. Therefore we can assume that $s_k \in J_{\alpha_k}$. By (c) from Lemma 1 and the remarks above,

$$\mathrm{Dp}(g(a_i), f) = \mathrm{Dp}(k_1 a_{s_1} + \cdots + k_n a_{s_n}, f) = \min_k \mathrm{Dp}(s_k, h) = \min_k \alpha_k.$$

On the other hand, by Lemma 2, as g commutes with f,

$$\alpha \leq \mathrm{Dp}(a_i, f) \leq \mathrm{Dp}(g(a_i), f).$$

Combining both, we get $\alpha \leq \alpha_k$ for $k = 1, \ldots, n$. Hence $g(a_i) \in \mathrm{Cl}\{a_i \mid i \in J_\beta,\ \alpha \leq \beta < \alpha_0\}$, so we have proved (b).

(c) Extend g to a function $g_1\colon \{a_i \mid i \in I\} \to A'$ in the following way: if $g(a_t) = k_1 a_{t_1} + \cdots + k_n a_{t_n}$, then let

$$g_1(a_{\langle 0, t, \eta\rangle}) = k_1 a_{\langle 0, t_1, \eta\rangle} + \cdots + k_n a_{\langle 0, t_n, \eta\rangle},$$

$$g_1(a_{\langle a,t,m\rangle}) = k_1 a_{\langle 1,t_1,m\rangle} + \cdots + k_n a_{\langle 1,t_2,m\rangle},$$
$$g_1(a_s) = a_s \text{ for } s \in I_0.$$

It is easy to check that g_1 is well defined (because $t \in J_\alpha$ and $t_k \in J_\beta$ imply $\alpha \leq \beta$) and it has a unique extension to $g_2 \in \mathrm{End}(A')$. In order to check that g_2 and f commute, it suffices to prove that for every $s \in I$

$$f \circ g_2(a_s) = g_2 \circ f(a_s),$$

and this is quite easy. □

A linear combination $\tau(x_1, \ldots, x_n) = k_1x_1 + \cdots + k_nx_n$ is called *beautiful* (see [24], where similar terms were called *beautiful terms*) if

1. we have $\tau(\tau(x_1^1, \ldots, x_n^1), \tau(x_1^2, \ldots, x_n^2), \ldots \tau(x_1^n, \ldots, x_n^n)) = \tau(x_1^1, \ldots, x_n^n)$;
2. we have $\tau(x, \ldots, x) = x$.

The condition (2) implies $k_1 + \cdots + k_n = 1$. It is clear that the condition (1) implies $k_ik_j = \delta_{ij}k_i$ for all $i, j = 1, \ldots, n$.

We can see that all k_j, except one (let it be k_i), are equal to zero, and $k_i = 1$, i.e., all beautiful linear combinations have the form x_i for some $i \in I$. Therefore the following lemma is trivial.

Lemma 4. *The set of beautiful linear combinations is closed under substitution.*

Theorem 1. *There is a formula $\tilde{\varphi}(\ldots)$ such that the following holds. Let $\{f_i\}_{i\in\mu}$ be a set of elements of* $\mathrm{End}(A')$*. Then we can find a vector $\bar{g}$ such that the formula $\tilde{\varphi}(f, \bar{g})$ holds in* $\mathrm{End}(A')$ *if and only if $f = f_i$ for some $i \in \mu$.*

Proof. Suppose that a set $\{a_i \mid i \in I^*\} \subset A$ is independent, consists of elements of the same order, and $\mathrm{Cl}\{a_i \mid i \in I^*\} = A'$. Let $J \subseteq I^*$ and $\mu = |J| = |I^* \setminus J|$. For notational simplicity let $J = \{\langle\alpha, \beta\rangle \mid \alpha, \beta \in \mu\}$, $a_{\langle\alpha,\beta\rangle} = a_\alpha^\beta$.

Lemma 5. *There is a formula $\varphi(f)$ with one free variable f such that $\varphi(f)$ holds in* $\mathrm{End}(A')$ *if and only if there is an ordinal number $\alpha \in \mu$ such that for all $\beta \in \mu$*

$$f(a_0^\beta) = a_\alpha^\beta.$$

Now we shall show the proof of Theorem 1 with the help of Lemma 5.

Let a function $f_0^*\colon A' \to A'$ map the set $\{a_0^\alpha \mid \alpha \in \mu\}$ onto the set $\{a_t \mid t \in I^*\}$, and let $f_0^*(a_\alpha^\beta) = f_0^*(a_0^\beta)$. Suppose that we have a set $\{f_i\}_{i\in\mu}$ and let the function $f^*\colon A' \to A'$ be such that

$$f^*(a_\alpha^\beta) = f_\alpha \circ f_0^*(a_\alpha^\beta).$$

Let $f_1^*\colon A \to A$ map the set $\{a_t \mid t \in I^*\}$ onto the set $\{a_0^\beta \mid \beta \in \mu\}$. Let the formula $\tilde{\varphi}(f', \ldots)$ say that there exists $f \in \mathrm{End}(A')$ such that

1. $\varphi(f)$;
2. $f' \circ f_0^* \circ f_1^* = f^* \circ f \circ f_1^*$.

Then $\vDash \varphi(f)$ if and only if there exists $\alpha \in \mu$ such that

$$\forall \beta \in \mu \; f(a_0^\beta) = a_\alpha^\beta.$$

Therefore

$$f' \circ f_0^* \circ f_1^*(a_t) = f^* \circ f \circ f_1^*(a_t) \Leftrightarrow f' \circ f_0^*(a_0^\beta) = f^* \circ f(a_0^\beta)$$

$$\Leftrightarrow f' \circ f_0^*(a_0^\beta) = f^*(a_\alpha^\beta) \Leftrightarrow f' \circ f_0^*(a_0^\beta) = f_\alpha \circ f_0^*(a_0^\beta).$$

Let $f_0^*(a_0^\beta) = a_{t_\beta}$. Then

$$f'(a_{t_\beta}) = f_\alpha(a_{t_\beta}),$$

what we needed. $\square$

Proof of Lemma 5. We partition our proof to three cases.

Case I. $\mu = \omega$.

Introduce the following mappings

1. $f_2^* \in \mathrm{End}(A')$ such that $f_2^*(a_t) = a_0^0$ for all $t \in I^*$;
2. $f_3^* \in \mathrm{End}(A')$ such that $t \in I^* \setminus J \Rightarrow f_3^*(a_t) = a_0^0$, $t \in J \Rightarrow f_3^*(a_n^m) = a_{n+1}^m$;
3. $f_4^* \in \mathrm{End}(A')$ such that $t \in I^* \setminus J \Rightarrow f_4^*(a_t) = a_0^0$, $t \in J \Rightarrow f_4^*(a_n^m) = a_n^{m+1}$;
4. $f_5^* \in \mathrm{End}(A')$ such that $t \in I^* \setminus J \Rightarrow f_5^*(a_t) = a_0^0$, $t \in J \Rightarrow f_5^*(a_n^m) = a_m^n$;
5. $f_6^* \in \mathrm{End}(A')$ such that $t \in I^* \setminus J \Rightarrow f_6^*(a_t) = a_0^0$, $t \in J \Rightarrow f_6^*(a_n^m) = a_n^n$;
6. $f_7^* \in \mathrm{End}(A')$ such that $t \in I^* \setminus J \Rightarrow f_7^*(a_t) = a_0^0$, $t \in J \Rightarrow f_7^*(a_n^m) = a_n^0$.

Suppose that

$$\begin{aligned}
B_0 &= \mathrm{Cl}\{a_0^0\},\\
B_1 &= \mathrm{Cl}\{a_n^0 \mid n \in \omega\},\\
B_2 &= \mathrm{Cl}\{a_n^m \mid n, m \in \omega\},\\
B_3 &= \mathrm{Cl}\{a_0^m \mid m \in \omega\}.
\end{aligned}$$

Introduce now the following first order formulas.

1. $\varphi_1(f; \dots)$ says that the restriction of the function f on the set B_1 is a function with the image in B_1, commuting with $f_3^*|_{B_1}$. As a formula we have the following:

   $$\varphi_1(f; \dots) = (\rho_{B_1} \circ f \circ \rho_{B_1} = f \circ \rho_{B_1}) \wedge (f \circ \rho_{B_1} \circ f_3^* \circ \rho_{B_1} = f_3^* \circ \rho_{B_1} \circ f \circ \rho_{B_1}),$$

   where ρ_{B_1} is the projection onto B_1.
2. Similarly, $\varphi_2(f; \dots)$ says that the formula $\varphi_1(f; \dots)$ holds and the image of $f|_{B_2}$ is included in B_2 and f_{B_2} commutes with $f_4^*|_{B_2}$.
3. The formula $\varphi_3(f; \dots)$ says that the formula $\varphi_2(f; \dots)$ holds and $(f_5^* \circ f \circ f_5^* \circ f)|_{B_0} = (f_6^* \circ f)|_{B_0}$.
4. The formula $\varphi_4(f; \dots)$ says that the formula $\varphi_3(f; \dots)$ holds and $(f_2^* \circ f)|_{B_0} = f_2^*|_{B_0}$.
5. The formula $\varphi_5^*(f; \dots)$ says that there exists $f' \in \mathrm{End}(A')$ such that $f'|_{B_3} = f|_{B_3}$ and the formula $\varphi_4(f'; \dots)$ holds.

Now we note the following.

1. The formula $\varphi_1(f; \dots)$ holds if and only if

$$f(a_n^0) = q_1 a_{n+l_1}^0 + \dots + q_k a_{n+l_k}^0$$

for some $q_1, \dots, q_k \in \mathbb{Z}$, $l_1, \dots, l_k \in \omega$ for any $n \in \omega$.

Actually, $f\colon B_1 \to B_1$ means that $f(a^0_n) = s_1 a^0_{n_1} + \cdots + s_k a^0_{n_k}$, and $f|_{B_1} \circ f^*_3|_{B_1} = f^*_3|_{B_1} \circ f|_{B_1}$ is equivalent to the condition that for any $n \in \omega$

$$f \circ f^*_3(a^0_n) = f^*_3 \circ f(a^0_n).$$

Let $f(a^0_0) = r_1 a_{l_1} + \cdots + r_k a_{l_k}$. Then $f(a^0_1) = f^*_3 \circ f(a^0_0) = r_1 a_{1+l_1} + \cdots + r_k a_{1+l_k}$ and so on by induction.

2. The formula $\varphi_2(f;\dots)$ holds if and only if

$$f(a^m_n) = r_1 a^m_{n+l_1} + \cdots + r_k a^m_{n+l_k}.$$

Actually, from $\varphi_1(f)$ it follows that for any $n \in \omega$

$$f(a^0_n) = r_1 a^0_{n+l_1} + \cdots + r_k a^0_{n+l_k},$$

and

$$f(a^1_n) = f \circ f^*_4(a^0_n) = f^*_4 \circ f(a^0_n) = f^*_4(r_1 a^0_{n+l_1} + \cdots + r_k a^0_{n+l_k}) = r_1 a^1_{n+l_1} + \cdots + r_k a^1_{n+l_k},$$

and so on by induction.

3. The formula $\varphi_3(f;\dots)$ holds if and only if the formula $\varphi_2(f;\dots)$ holds, where $l_1,\dots,l_k$ are distinct and the corresponding linear combination τ satisfies the condition (1) of the definition of beautiful linear combination.

Actually, let the formula $\varphi_2(f)$ be true and τ be the corresponding linear combination. Then

$$\begin{aligned} & f^*_5 \circ f \circ f^*_5 \circ f|_{B_0} = f^*_6 \circ f|_{B_0} \Leftrightarrow f^*_5 \circ f \circ f^*_5 \circ f(a^0_0) = f^*_6 \circ f(a^0_0) \\ & \quad \Leftrightarrow f^*_5 \circ f \circ f^*_5(\tau(a^0_{l_1},\dots,a^0_{l_k})) = f^*_6(\tau(a^0_{l_1},\dots,a^0_{l_k})) \Leftrightarrow f^*_5 \circ f(\tau(a^{l_1}_0,\dots,a^{l_k}_0)) = \tau(a^{l_1}_{l_1},\dots,a^{l_k}_{l_k}) \\ & \quad \Leftrightarrow f^*_5(\tau(\tau(a^{l_1}_{l_1},\dots,a^{l_k}_{l_1}),\dots,\tau(a^{l_1}_{l_k},\dots,a^{l_k}_{l_k})) = \tau(a^{l_1}_{l_1},\dots,a^{l_k}_{l_k}) \\ & \quad \Leftrightarrow \tau(\tau(a^{l_1}_{l_1},\dots,a^{l_1}_{l_k}),\dots,\tau(a^{l_k}_{l_1},\dots,a^{l_k}_{l_k})) = \tau(a^{l_1}_{l_1},\dots,a^{l_k}_{l_k}). \end{aligned}$$

It is equivalent to the condition (1) from the definition of linear combination. The converse condition is proved in the same manner.

4. The formula $\varphi_4(f;\dots)$ holds if and only if

$$f(a^m_n) = \tau(a^m_{n+l_1},\dots,a^m_{n+l_k}),$$

where $\tau(x_1,\dots,x_k)$ is a beautiful linear combination, i.e., $f(a^m_n) = a^m_{n+l_s}$.

Since $\varphi_4(f) \Rightarrow \varphi_3(f)$, we only need to show that $a^0_0 = \tau(a^0_0,\dots,a^0_0)$.

Actually,

$$f^*_2 \circ f|_{B_0} = f^*_2|_{B_0} \Leftrightarrow f^*_2 \circ f(a^0_0) = f^*_2(a^0_0) \Leftrightarrow f^*_2(\tau(a^0_{l_1},\dots,a^0_{l_k})) = a^0_0 \Leftrightarrow \tau(a^0_0,\dots,a^0_0) = a^0_0.$$

5. The formula $\varphi_5(f)$ holds if and only if

$$f(a^n_0) = \tau(a^n_{l_1},\dots,a^n_{l_k})$$

for some beautiful linear combination τ and $l_1,\dots,l_k \in \omega$ for all $n \in \omega$.

This follows immediately from $f^*_5 \circ f^*_7 \circ f^*_5(a^m_0) = f^*_5 \circ f^*_7(a^0_m) = a^m_0$.

Case II. The cardinal number $\mu = |J|$ is regular and $\mu > \omega$.

Let $I^* \setminus J = I_0 \cup \{\langle \alpha,\delta,n\rangle \mid \alpha \in \mu,\ \delta \in \mu,\ \mathrm{cf}\delta = \omega,\ n \in \omega\}$, $|I_0| = \mu$, and let us denote $a^{\beta,n}_\alpha = a_{\langle \alpha,\beta,n\rangle}$.

For every limit ordinal $\delta \in \mu$ such that $\mathrm{cf}\delta = \omega$, choose an increasing sequence $(\delta_n)_{n\in\omega}$ of ordinals less than δ such that their limit is δ and for each $\beta \in \mu$, $n \in \omega$ the set $\{\delta \in \mu \mid \beta = \delta_n\}$ is a stationary subset in μ (see [25]).

Let us define some f^*_i by defining $f^*_i(a_t)$ for some $t \in I^*$ and understanding that when $f^*_i(a_t)$ is not explicitly defined, it is a^0_0.

So let for $\alpha, \beta \in \mu$

$$f^*_2(a^\beta_\alpha) = a^0_0; \quad f^*_3(a^\beta_\alpha) = a^0_\alpha; \quad f^*_4(a^\beta_\alpha) = a^\beta_0; \quad f^*_5(a^\beta_\alpha) = a^\alpha_\beta; \quad f^*_6(a^\beta_\alpha) = a^\alpha_\alpha;$$

let for $\delta \in \mu$, $\mathrm{cf}\delta = \omega$

$$f_7^*(a_\alpha^\delta) = a_\alpha^{\delta,0};$$

and let for $\delta \in \mu$, $\mathrm{cf}\delta \neq \omega$

$$f_7^*(a_\alpha^\delta) = a_\alpha^\delta;$$

and, further,

$$f_8^*(a_\alpha^\beta) = a_\alpha^\beta, \quad f_8^*(a_\alpha^{\delta,n}) = a_\alpha^{\delta,n+1};$$
$$f_9^*(a_\alpha^\beta) = a_\alpha^\beta, \quad f_9^*(a_\alpha^{\delta,n}) = a_\alpha^{\delta_n}.$$

Let

$$\begin{aligned}
B_0 &= \mathrm{Cl}\{a_0^0\},\\
B_1 &= \mathrm{Cl}\{a_\alpha^0 \mid \alpha \in \mu\},\\
B_2 &= \mathrm{Cl}\{a_0^\beta \mid \beta \in \mu\},\\
B_3 &= \mathrm{Cl}\{a_\alpha^\beta \mid \alpha, \beta \in \mu\},\\
B_4 &= \mathrm{Cl}\{a_\alpha^\beta, a_\alpha^{\beta,n} \mid \alpha, \beta \in \mu,\ n \in \omega\},\\
B_5 &= \mathrm{Cl}\{a_0^\beta, a_0^{\beta,n} \mid \beta \in \mu,\ n \in \omega\},\\
B_6 &= \mathrm{Cl}\{a_0^\beta \mid \beta \in \mu,\ \mathrm{cf}\beta = \omega\},\\
B_7 &= \mathrm{Cl}\{a_\alpha^\beta \mid \alpha, \beta \in \mu,\ \mathrm{cf}\beta = \omega\}.
\end{aligned}$$

Clearly f_3^*, f_4^*, and f_9^* are projections onto B_1, B_2, and B_3, respectively. Let f_{10}^*, f_{11}^*, and f_{12}^* be projections onto B_4, B_5, and B_6, respectively.

Now we apply Lemma 3 with $J = \{\langle \alpha, \beta \rangle \mid \alpha, \beta \in \mu\}$, $J_\beta = \{\langle \alpha, \beta \rangle \mid \alpha \in \mu\}$, $I = I^*$, and $f = f_{13}^* \in \mathrm{End}(A')$.

Let the first order formula $\varphi^1(f, g; \ldots)$ say that

1. f and g are conjugate to f_2^*;
2. $\mathrm{Rng}\, f, \mathrm{Rng}\, g \subset B_3$;
3. $\exists h \in \mathrm{End}(A')$ $(h \circ f_{13}^* = f_{13}^* \circ h \wedge \mathrm{Rng}\, h|_{B_3} \subseteq B_3 \wedge h \circ f = g)$.

We shall write $\varphi^1(f, g; \ldots)$ also in the form $f \leq g$.

If f and g are conjugate to f_2^*, $f(a_0^0) = a_\alpha^\beta$, and $g(a_0^0) = \tau(a_{\alpha_1}^{\beta_1}, \ldots, a_{\alpha_k}^{\beta_k})$, then $f \leq g$ if and only if $\beta \leq \beta_1, \ldots, \beta \leq \beta_k$, which is easy to see from Lemma 3.

Let the first order formula $\varphi_2(f)$ say that

1. $\mathrm{Rng}\, f|_{B_2} \subseteq B_3$; $\mathrm{Rng}\, f|_{B_5} \subseteq B_4$; $\mathrm{Rng}\, f|_{B_0} \subseteq B_1$; $\mathrm{Rng}\, f|_{B_6} \subseteq B_7$;
2. for any $g \in \mathrm{End}(A')$ conjugate to f_2^*, if $\mathrm{Rng}\, g \subseteq B_2$, then $g \leq f \circ g$;
3. $f|_{B_5}$ commutes with f_7^*, f_8^*, and f_9^*;
4. $f|_{B_3}$ commutes with f_3^*.

Statement 1. *The formula $\varphi_2(f)$ holds in $\mathrm{End}(A')$ if and only if for any $\beta \in \mu$*

$$f(a_0^0) = \tau(a_{\alpha_1}^\beta, \ldots, a_{\alpha_k}^\beta) \quad \textit{and} \quad f(a_0^{\beta,n}) = \tau(a_{\alpha_1}^{\beta,n}, \ldots, a_{\alpha_k}^{\beta,n})$$

for some linear combination τ and ordinal numbers $\alpha_1, \ldots, \alpha_k \in \mu$ (which do not depend on β).

Proof. Assume that $\varphi_2(f)$ holds for a given function f, and let

$$f(a_0^\beta) = \tau_\beta(a_{\alpha_{\beta,1}}^{\gamma_{\beta,1}}, \dots, a_{\alpha_{\beta,k_\beta}}^{\gamma_{\beta,k_\beta}}).$$

Without loss of generality we can assume that the ordinal numbers $\alpha_{\beta,i}$ grow as i grows.

Choose g such that $g(a_t) = a_0^\beta$ for all $t \in I^*$. Then g is conjugate to f_2^*, $\operatorname{Rng} g \subseteq B_2$, and therefore, by (2), $g \leq f \circ g$. Since $g(a_0^0) = a_0^\beta$ and $f(a_0^0) = \tau_\beta(a_{\alpha_{\beta,1}}^{\gamma_{\beta,1}}, \dots, a_{\alpha_{\beta,k_\beta}}^{\gamma_{\beta,k_\beta}})$, we have that $\beta \leq \gamma_{\beta,i}$ for all $i = 1, \dots, k_\beta$.

Since the cardinal number μ is regular and $\mu > \omega$, we have that for any $\beta_0 < \mu$

$$\sup\{\gamma_{\beta,i} \mid i = 1, \dots, k_\beta;\ \beta \in \beta_0\} < \mu.$$

Hence the set

$$S = \{\beta_0 \mid \beta_0 \in \mu \wedge \forall \beta\, (\beta \in \beta_0 \wedge 1 \leq i \leq k_\beta \Rightarrow \gamma_{\beta,i} \in \beta_0)\}$$

is an unbounded subset in μ; and by its definition it is closed (indeed, $\beta_1 = \sup\{\gamma_{\beta,i} \mid i = 1, \dots, k_\beta,\ \beta \in \beta_0\}$, $\beta_2 = \sup\{\gamma_{\beta,i} \mid i = 1, \dots, k_\beta,\ \beta \in \beta_1\}$, ..., $\bar\beta = \bigcup_{l \in \omega} \beta_l$, whence $\bar\beta < \mu$, $\bar\beta \in S$, and so on).

Now we shall prove that for $\delta \in S$, $\mathrm{cf}\delta = \omega$ implies $\gamma_{\delta,i} = \delta$. Suppose that $\gamma = \gamma_{\delta,i_0} \neq \delta$ for some i_0, which yields $\delta < \gamma_{\delta,i_0}$ as was said above. As δ_n is increasing (as a function of n) and its limit is δ, for some $n \in \omega$ big enough,

$$\delta < \gamma_{\delta,i_0}(n) \tag{2}$$

and $\gamma_{\delta,i_1} \neq \gamma_{\delta,i_2} \Rightarrow \gamma_{\delta,i_1}(n) \neq \gamma_{\delta,i_2}(n)$.

Since $\operatorname{Rng} f|_{B_6} \subseteq B_7$, we see that necessarily $\gamma_{\delta,i}$ has cofinality ω, and as $f|_{B_5}$ commutes with f_7^*, we see that

$$f(a_0^{\delta,0}) = f \circ f_7^*(a_0^\delta) = f_7^* \circ f(a_0^\delta) = \tau_\delta(a_{\alpha_{\delta,1}}^{\gamma_{\delta,1},0}, \dots, a_{\alpha_{\delta,s}}^{\gamma_{\delta,s},0}).$$

As $f|_{B_5}$ commutes with f_8^*,

$$f(a_0^{\delta,n}) = f \circ f_8^*(a_0^{\delta,n-1}) = f_8^* \circ f(a_0^{\delta,n-1}),$$

whence

$$f(a_0^{\delta,n}) = \tau_\delta(a_{\alpha_{\delta,1}}^{\gamma_{\delta,1},n}, \dots).$$

Since $f|_{B_5}$ commutes with f_9^*, we have

$$f(a_0^{\delta_n}) = \tau_\delta(a_{\alpha_{\delta,1}}^{\gamma_{\delta,1}(n)}, \dots)$$

and the linear combination $\tau_\delta(a_{\alpha_{\delta,1}}^{\gamma_{\delta,1}(n)}, \dots)$ is reduced.

But $f(a_0^{\delta_n}) = \tau_{\delta_n}(a_{\alpha_{\delta_n,1}}^{\gamma_{\delta_n,1}}, \dots)$, where the linear combination on the right-hand side of the equality is also reduced. Hence

$$\gamma_{\delta,i_0}(n) \in \{\gamma_{\delta_n,i} \mid i \leq i \leq k_{\delta_n}\},$$

but on the one hand, $\delta < \gamma_{\delta,i_0}(n)$ by (2), and on the other hand, $\delta < \delta_n \Rightarrow \gamma_{\delta_n,i} < \delta$, as $\delta \in S$, contradiction. Consequently, $\gamma_{\delta,i} = \delta$ for all $\delta \in S$.

We know that for all $\beta \in \mu$ and $n \in \omega$ the set $\{\delta \in \mu \mid \mathrm{cf}\delta = \omega \wedge \delta_n = \beta\}$ is stationary, so there is $\delta \in S$, where $\mathrm{cf}\delta = \omega$, such that $\delta_n = \beta$.

As before, we can show that

$$f(a_0^{\delta_n}) = \tau_\delta(a_{\alpha_{\delta,1}}^{\delta_n}, \dots, a_{\alpha_{\delta,k_\delta}}^{\delta_n}) = \tau_{\delta_n}(a_{\alpha_{\delta_n,1}}^{\gamma_{\delta_n,1}}, \dots, a_{\alpha_{\delta_n,k_{\delta_n}}}^{\gamma_{\delta_n,k_{\delta_n}}}),$$

where the linear combination τ_{δ_n} is reduced. Therefore $\gamma_{\delta_n,i} \in \{\delta_n\}$ for all i, so $\gamma_{\delta_n,i} = \delta_n$, i.e., $\gamma_{\beta,i} = \delta_n$ for all i. As the linear combination $\tau_\beta(a_{\alpha_{\beta,1}}^{\gamma_{\beta,1}}, \dots)$ is reduced, we have that the ordinals $\alpha_{\beta,i}$, where $1 \leq i \leq k_\beta$, are distinct.

As $f|_{B_3}$ commutes with f_3^*, for every β

$$\tau_\beta(a_{\alpha_{\beta,1}}^0, \dots, a_{\alpha_{\beta,k_\beta}}^0) = \tau_0(a_{\alpha_{0,1}}^0, \dots, a_{\alpha_{\alpha_0,k_0}}^0).$$

As the $\alpha_{\beta,i}$ are distinct, necessarily

$$\{\alpha_{\beta,i} \mid 1 \le i \le k_\beta\} = \{\alpha_{0,i} \mid 1 \le i \le k_0\},$$

but as $\alpha_{\beta,i}$ is increasing with i (for each β, by the choice of $\alpha_{\beta,i}$), necessarily $\alpha_{\beta,i} = \alpha_{0,i}$ and $k_\beta = k_0$, i.e., $\tau = \tau_0$.

Therefore $f(a_0^{\beta,n}) = \tau_0(a_{\alpha_{0,1}}^{\beta,n}, \dots)$.

The other direction of this assertion is immediate. □

Let $\varphi_3(f)$ say that

1. $\mathrm{Rng}\, f|_{B_2} \subseteq B_3$;
2. $\exists f_1 \in \mathrm{End}(A')\ (f_1|_{B_2} = f|_{B_2} \wedge \varphi_2(f_1))$.

The formula $\varphi_3(f)$ holds if and only if

$$f(a_0^\beta) = \tau(a_{\gamma_1}^\beta, \dots, a_{\gamma_k}^\beta)$$

for any $\beta \in \mu$ and some $\tau, \gamma_1, \dots, \gamma_k$. This follows immediately from Statement 1.

Let the formula $\varphi_4(f)$ say that

1. $\mathrm{Rng}\, f|_{B_2} \subseteq B_3$;
2. $\varphi_3(f)$;
3. $\forall g \in \mathrm{End}(A')\ (\varphi_3(g) \Rightarrow g \circ f_5 \circ f|_{\tilde{a}_0^0} = f_5 \circ f \circ f_5 \circ g|_{\tilde{a}_0^0} \wedge f_5^* \circ f \circ f_5^* \circ f|_{\tilde{a}_0^0} = f_6^* \circ f|_{\tilde{a}_0^0} \wedge f_2^* \circ f|_{\tilde{a}_0^0} = f_2^*|_{\tilde{a}_0^0}$.

As in case I, we can check that the formula $\varphi_4(f)$ holds in $\mathrm{End}(A')$ if and only if

$$f(a_0^\beta) = \tau(a_{\gamma_1}^\beta, \dots, a_{\gamma_k}^\beta),$$

where τ is a beautiful linear combination.

Case III. μ is a singular cardinal number.

Let $\mu_1 < \mu$, where μ_1 is a regular cardinal number and $\mu_1 > \omega$. Let $I^* \setminus J = I_0 \cup \{\langle \alpha, \delta, n\rangle \mid \alpha \in \mu,\ \delta \in \mu_1,\ \mathrm{cf}\delta = \omega,\ n \in \omega\}$, $|I_0| = \mu$, $a_\alpha^{\beta,n} = a_{\langle \alpha,\beta,n\rangle}$.

For every limit ordinal $\delta \in \mu_1$ such that $\mathrm{cf}\delta = \omega$, similarly to the previous case, choose an increasing sequence $(\delta_n)_{n\in\omega}$ of ordinal numbers less than δ, with limit δ, such that for any $\beta \in \mu_1$ and $n \in \omega$ the set $\{\delta \in \mu_1 \mid \beta = \delta_n\}$ is a stationary subset in μ_1.

Let

$$\begin{aligned} B_1 &= \mathrm{Cl}\{a_\alpha^0 \mid \alpha \in \mu\},\\ B_2 &= \mathrm{Cl}\{a_0^\beta \mid \beta \in \mu_1\},\\ B_3 &= \mathrm{Cl}\{a_\alpha^\beta \mid \alpha \in \mu,\ \beta \in \mu_1\}. \end{aligned}$$

As in case II, we can define the functions f_i^* in such a way that for some $\varphi^0(\dots)$ the formula $\varphi^0(f;\dots)$ holds in $\mathrm{End}(A')$ if and only if there exist a linear combination τ and distinct ordinal numbers $\alpha_1, \dots, \alpha_n \in \mu$ such that for every $\beta \in \mu_1$

$$f(a_0^\beta) = \tau(a_{\alpha_1}^\beta, \dots, a_{\alpha_n}^\beta).$$

Let the formula $\varphi^1(f)$ say that

1. $\mathrm{Rng}\, f|_{\tilde{a}_0^0} \subseteq B_2$;
2. for every $g \in \mathrm{End}(A')$ $\varphi^0(g) \Rightarrow (f \circ g)|_{\tilde{a}_0^0} = (g \circ f)|_{\tilde{a}_0^0}$.

It is easy to check that the formula $\varphi^1(f)$ holds if and only if there exist a linear combination σ and distinct ordinal numbers $\beta_1, \dots, \beta_m \in \mu_1$ such that for any $\alpha \in \mu$

$$f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_m}_\alpha).$$

As the cardinal number μ_1 is regular, we can use case II. Thus, there is a formula $\varphi^2(f; \dots)$ such that $\varphi^2(f)$ holds in $\mathrm{End}(A')$ if and only if there exist a beautiful linear combination σ and distinct $\beta_i \in \mu_1$ such that for any $\alpha \in \mu$

$$f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_m}_\alpha).$$

Let $\mu = \bigcup\limits_{i \in \mathrm{cf}\mu} \mu_i$, where $\mu_i \in \mu$ and the sequence (μ_i) increases. We just prove that for every $\gamma \in \mathrm{cf}\mu$ there is a function $\bar{f}^*_\gamma$ such that

1. the formula $\varphi^2[f, \bar{f}^*_\gamma]$ holds in $\mathrm{End}(A')$ if and only if there exist a beautiful linear combination σ and distinct $\beta_i \in \mu^+_\gamma$ such that for all $\alpha \in \mu$
$$f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_m}_\alpha);$$
2. $f^*_{\gamma\,0}$ is a projection onto
$$\mathrm{Cl}\{a^\beta_\alpha \mid \alpha \in \mu,\ \beta \in \mu^+_\gamma\}.$$

Choose $\mu_1 = \mu^+_\gamma$ and consider the same $\varphi^2(f)$ as in the case II, with $f^*_{\gamma\,0}$ being a projection onto $\mathrm{Cl}\{a^\beta_\alpha \mid \alpha \in \mu,$ $\beta \in \mu_1\}$.

Let τ and σ be beautiful linear combinations, $\beta_1, \dots, \beta_m \in \mu^+_{\gamma_k}$, $k = 1, \dots, n$, and for every $\alpha \in \mu$

$$f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_n}_\alpha).$$

We show that in this case we have

$$\varphi^2(f, \tau(\bar{f}^*_{\gamma_1}, \dots, \bar{f}^*_{\gamma_n})).$$

The formula $\varphi^2(f, \tau(\bar{f}^*_{\gamma_1}, \dots, \bar{f}^*_{\gamma_n}))$ holds if and only if $f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_m}_\alpha)$ for all $\alpha \in \mu$ and distinct $\beta_1, \dots, \beta_n \in \mu_{\gamma_1}$, which is true. Further, we have that $\tau(f^*_{\gamma_1 0}, \dots, f^*_{\gamma_n 0})$ is a projection onto the set $\mathrm{Cl}\{a^\beta_\alpha \mid \alpha \in \mu,$ $\beta \in \mu^+_{\gamma_1}\}$ because

$$\tau(x_1, \dots, x_n) = x_s.$$

Recall how we proved Theorem 1 from Lemma 5. This proof easily implies that there exist a formula φ^3 and a vector of functions g^* such that the formula $\varphi^3(\bar{f}, \bar{g}^*)$ holds if and only if $\bar{f} = \bar{f}^*_\gamma$ for some $\gamma \in \mu$.

Let now the formula $\varphi^4(f, \bar{g}^*)$ say that there exists $\bar{f}_1$ such that $\varphi^3(\bar{f}_1, \bar{g}^*)$ and for every $\bar{f}_2$ satisfying the formulas $\varphi^3(\bar{f}_2, \bar{g}^*)$ and $\mathrm{Rng}(\bar{f}_1)_0 \subseteq \mathrm{Rng}(\bar{f}_2)_0$, also $\varphi^2(f, \bar{f}_2)$ holds. If the formula $\varphi^4(f, g^*)$ holds, then there exists $\bar{f}_1 = \bar{f}^*_\gamma$ for some $\gamma \in \mu$, and for every $\bar{f}_2 = \bar{f}^*_\lambda$ (where $\lambda \geq \gamma$) we have the formula

$$f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_m}_\alpha),$$

where $\beta_1, \dots, \beta_m < \mu^+_\lambda$.

Let f be such that

$$f(a^0_\alpha) = \sigma(a^{\beta_1}_\alpha, \dots, a^{\beta_m}_\alpha), \quad \beta_1, \dots, \beta_m \in \mu.$$

Then $\beta_1, \dots, \beta_m \in \mu^+_\gamma$ for $\gamma \in \mathrm{cf}\mu$ and therefore the formula $\varphi^4(f, g^*)$ holds for some g^*.

Now we only need to consider the formula $\varphi^4(f^*_5 \circ f \circ f^*_5)$, which is the required formula. $\square$

4 Formulation of the Main Theorem, Converse Theorems, Different Cases

4.1 Second Order Language of Abelian Groups

As we mentioned above, we shall consider second order models of Abelian groups, i.e., consider the second order group language, where the 3-place symbol will denote not multiplication, but addition (i.e., we shall write $x_1 = x_2 + x_3$ instead of $P^3(x_1, x_2, x_3)$).

As we see, formulas $\varphi(\ldots)$ of the language $\mathcal{L}_2$ consist of the following subformulas:

1. $\forall x$ ($\exists x$);
2. $x_1 = x_2$ and $x_1 = x_2 + x_3$, where every variable x_1, x_2, and x_3 either is a free variable of the formula φ or is defined in the formula φ with the help of the subformulas $\forall x_i$ or $\exists x_i$, $i = 1, 2, 3$;
3. $\forall P(v_1, \ldots, v_n)$ ($\exists P(v_1, \ldots, v_n)$), $n > 0$;
4. $P(x_1, \ldots, x_n)$, where every variable $x_1, \ldots, x_n$, and also every "predicate" variable $P(v_1, \ldots, v_n)$ either is a free variable of the formula φ, or is defined in this formula with the help of the subformula $\forall x_i$, $\exists x_i$, $\forall P(v_1, \ldots, v_n)$, $\exists P(v_1, \ldots, v_n)$.

Equivalence of two Abelian groups A_1 and A_2 in the language $\mathcal{L}_2$ will be denoted by

$$A_1 \equiv_{\mathcal{L}_2} A_2, \quad \text{or} \quad A_1 \equiv_2 A_2.$$

As we remember, the *theory* of a model $\mathcal{U}$ of a language $\mathcal{L}$ is the set of all sentences of the language $\mathcal{L}$ which are true in this model. In some cases we shall consider together with theories $\mathrm{Th}_2(A) = \mathrm{Th}_{\mathcal{L}_2}(A)$ also theories $\mathrm{Th}_2^{\varkappa}(A)$, which contain those sentences φ of the language $\mathcal{L}_2$ that are true for arbitrary sequence

$$\langle a_1, \ldots, a_q, b_1^{l_1}, \ldots, b_s^{l_s} \rangle,$$

where $a_1, \ldots, a_q \in A$, $b_i^{l_i} \subset A^{l_i}$, and $|b_i^{l_i}| \leq \varkappa$. If $\varkappa \geq |A|$, then $\mathrm{Th}_2(A)$ and $\mathrm{Th}_2^{\varkappa}(A)$ coincide.

4.2 Formulation of the Main Theorem

If $A = D \oplus G$, where the group D is divisible and the group G is reduced, then the *expressible rank of the group A* is the cardinal number

$$r_{\exp} = \mu = \max(\mu_D, \mu_G),$$

where μ_D is the rank of D and μ_G is the rank of the basic subgroup of G.

We want to prove the following theorem.

Theorem 1. *For any infinite p-groups A_1 and A_2 elementary equivalence of endomorphism rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *implies coincidence of the second order theories* $\mathrm{Th}_2^{r_{\exp}(A_1)}(A_1)$ *and* $\mathrm{Th}_2^{r_{\exp}(A_2)}(A_2)$ *of the groups A_1 and A_2, bounded by the cardinal numbers $r_{\exp}(A_1)$ and $r_{\exp}(A_2)$, respectively.*

In Secs. 5–7, we shall separately prove this theorem for Abelian groups A_1 and A_2 with various properties and in Sec. 8 gather them and prove the main theorem.

Note that if the group A is finite, then the ring $\mathrm{End}(A)$ is also finite. Since in the case of finite models elementary equivalence (and also equivalence in the language $\mathcal{L}_2$) is equivalent to isomorphism of them, then in the case where one of the groups A_1 and A_2 is finite Theorem 1 follows from Theorem 11. Therefore we now suppose that the groups A_1 and A_2 are infinite.

4.3 Proofs of "Converse" Theorems

Let us prove two theorems which are, in some sense, converse to our main theorem.

Theorem 2. *For any Abelian groups A_1 and A_2, if the groups A_1 and A_2 are equivalent in the second order logic $\mathcal{L}_2$, then the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *are elementarily equivalent.*

Proof. Every 2-place predicate variable $P(v_1, v_2)$ will be called a *correspondence on the group A*. A correspondence $P(v_1, v_2)$ on the group A will be called a *function on the group A* (notation: $\mathrm{Func}(P(v_1, v_2))$, or simply $\mathrm{Func}(P)$) if it satisfies the condition

$$(\forall x\, \exists y\, P(x, y)) \wedge (\forall x\, \forall y_1\, \forall y_2\, P(x, y_1) \wedge P(x, y_2) \Rightarrow y_1 = y_2).$$

A function $P(v_1, v_2)$ will be called an *endomorphism on the group A* (notation: $\mathrm{Endom}(P(v_1, v_2))$, or simply $\mathrm{Endom}(P)$) if it satisfies the additional condition

$$\forall x_1\, \forall x_2\, \forall y_1\, \forall y_2\, P(x_1, y_1) \wedge P(x_2, y_2) \Rightarrow P(x_1 + x_2, y_1 + y_2).$$

Now consider an arbitrary sentence φ of the first order ring language. This sentence can contain the subformulas

1. $\forall x$;
2. $\exists x$;
3. $x_1 = x_2$;
4. $x_1 = x_2 + x_3$;
5. $x_1 = x_2 \cdot x_3$.

Let us translate this sentence to a sentence $\tilde{\varphi}$ of the second order group language by the following algorithm:

1. the subformula $\forall x\, (\ldots)$ is translated to the subformula
$$\forall P^x(v_1, v_2)\, (\mathrm{Endom}(P^x) \Rightarrow \ldots);$$
2. the subformula $\exists x\, (\ldots)$ is translated to the subformula
$$\exists P^x(v_1, v_2)\, (\mathrm{Endom}(P^x) \wedge \ldots);$$
3. the subformula $x_1 = x_2$ is translated to the subformula
$$\forall y_1\, \forall y_2\, (P^{x_1}(y_1, y_2) \Leftrightarrow P^{x_2}(y_1, y_2));$$
4. the subformula $x_1 = x_2 + x_3$ is translated to the subformula
$$\forall y\, \forall z_1\, \forall z_2\, \forall z_3\, (P^{x_2}(y, z_2) \wedge P^{x_3}(y, z_3) \Rightarrow (P^{x_1}(y, z_1) \Leftrightarrow z_1 = z_2 + z_3));$$
5. the subformula $x_1 = x_2 \cdot x_3$ is translated to the subformula
$$\forall y\, \forall z\, (P^{x_1}(y, z) \Rightarrow \exists t\, (P^{x_2}(y, t) \wedge P^{x_3}(t, z)).$$

We need to show that a sentence φ holds in the model $\mathrm{End}(A)$ if and only if the sentence $\tilde{\varphi}$ holds in the model A.

If A is a model of an Abelian group, then the model $\mathrm{End}(A)$ consists of sets of pairs of elements of the model A, $x = \{\langle u_1, u_2\rangle \mid u_1, u_2 \in A\}$, with the conditions

1. $\forall u_1 \, \exists u_2 \, \langle u_1, u_2 \rangle \in x$;

2. $\forall u_1 \, \forall u_2 \, \forall u_3 \, (\langle u_1, u_2 \rangle \in x \wedge \langle u_1, u_3 \rangle \in x \Rightarrow u_2 = u_3)$;

3. $\forall u_1 \, \forall u_2 \, \forall u_3 \, \forall u_4 \, (\langle u_1, u_3 \rangle \in x \wedge \langle u_2, u_4 \rangle \in x \Rightarrow \langle u_1 + u_2, u_3 + u_4 \rangle \in x)$.

Therefore a sequence $a_1, \ldots, a_q$ for which the formula φ is satisfied in the model $\mathrm{End}(A)$ is a sequence consisting of sets of pairs of elements from A satisfying the conditions (1)–(3).

Let us establish the identity bijection between elements of $\mathrm{End}(A)$ and the corresponding sets of pairs of the model A. Let an element a_i of the model $\mathrm{End}(A)$ correspond to a set $A_i \subset A \times A$.

1. If the formula φ has the form $x_i = x_j$, then φ holds for a sequence $a_1, \ldots, a_q$ if and only if $a_i = a_j$, i.e., a_i and a_j are equal endomorphisms of the model $\mathrm{End}(A)$, and the sets A_i and A_j consist of the same elements, i.e., in the model A for the sequence $A_1, \ldots, A_q$ the formula

$$\forall y_1 \, \forall y_2 \, (P^{x_i}(y_1, y_2) \Leftrightarrow P^{x_j}(y_1, y_2))$$

is true.

2. If the formula φ has the form $x_i = x_j + x_k$, then φ holds for a sequence $a_1, \ldots, a_q$ if and only if $a_i = a_j + a_k$, i.e., an endomorphism a_i is the sum of endomorphisms a_j and a_k, and this means that in the model A for every element $b \in A$ and for every $b_1, b_2, b_3 \in A$ such that $\langle b, b_1 \rangle \in A_i$, $\langle b, b_2 \rangle \in A_j$, $\langle b, b_3 \rangle \in A_k$, we have $b_1 = b_2 + b_3$ (i.e., formally speaking, $\langle b_1, b_2, b_3 \rangle \in I(Q_1^3)$). It is equivalent to $A \vDash \tilde{\varphi}$.

3. If the formula φ has the form $x_i = x_j \cdot x_k$, then the formula φ is true for a sequence $a_1, \ldots, a_q$ if and only if $a_i = a_j \cdot a_k$, i.e., the endomorphism a_i is a composition of endomorphisms a_j and a_k, and this means that in the model A for every $b_1 \in A$ and for every $b_2 \in A$ such that $\langle b_1, b_2 \rangle \in A_i$, there exists $b_3 \in A$ such that $\langle b_1, b_3 \rangle \in A_j$ and $\langle b_3, b_2 \rangle \in A_k$. This is equivalent to $A \vDash \tilde{\varphi}$.

4. If φ has the form $\theta_1 \wedge \theta_2$, θ_1 and θ_2 are true in the model $\mathrm{End}(A)$ for a sequence $a_1, \ldots, a_q$ if and only if $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are true in the model A for the sequence $A_1, \ldots, A_q$, then it is clear that the formula φ is true in the model $\mathrm{End}(A)$ for a sequence $a_1, \ldots, a_q$ if and only if the formula $\tilde{\varphi}$ is true in the model A for the sequence $A_1, \ldots, A_q$, because

$$\widetilde{\theta_1 \wedge \theta_2} = \tilde{\theta}_1 \wedge \tilde{\theta}_2.$$

5. A similar case is the formula φ having the form $\neg\theta$, because

$$\widetilde{\neg\theta} = \neg\tilde{\theta}.$$

6. Finally, suppose that the formula φ has the form $\forall x_i \, \psi$. The formula φ is true in the model $\mathrm{End}(A)$ for a sequence $a_1, \ldots, a_q$ if and only if the formula ψ is true in the model $\mathrm{End}(A)$ for the sequence $a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q$ for any $a \in \mathrm{End}(A)$, i.e., the formula $\tilde{\psi}$ is true in the model A for the sequence $A_1, \ldots, A_{i-1}, \bar{A}, A_{i+1}, \ldots, A_q$ for every set $\bar{A} \subset A \times A$ which is an endomorphism of the ring A, i.e., which satisfies the formula Endom. Therefore the formula φ is true in the model $\mathrm{End}(A)$ for a sequence $a_1, \ldots, a_q$ if and only if the formula

$$\widetilde{\forall x_i \, \psi} := \forall P^{x_i}(v_1, v_2) \, (\mathrm{Endom}(P^{x_i}) \Rightarrow \tilde{\psi})$$

is true for the sequence $A_1, \ldots, A_q$ in the model A.

Suppose now that Abelian groups A_1 and A_2 are equivalent in the language $\mathcal{L}_2$. Consider an arbitrary sentence φ of the first order ring language, which is true in $\mathrm{End}(A_1)$. Then the sentence $\tilde{\varphi}$ is true in the group A_1 and hence in the group A_2. Consequently, the sentence φ is true in the ring $\mathrm{End}(A_2)$. Therefore the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ are elementarily equivalent. $\square$

For the next theorem we need some formulas.

1. The formula

$$\mathrm{Gr}(P(v)) := \forall a \, \forall b \, (P(a) \wedge P(b) \Rightarrow \exists c \, (c = a + b \wedge P(c)) \wedge P(0) \wedge \forall a \, (P(a) \Rightarrow \exists b \, (b = -a \wedge P(b)))$$

holds for those sets $\{a \in A \mid P(a)\}$ that are subgroups in A, and only for them.

2. The formula

$$\mathrm{Cycl}(P(v)) := \mathrm{Gr}(P(v)) \wedge \exists a\, (P(a) \wedge \forall P_a(v)\, (\mathrm{Gr}(P_a(v)) \wedge P_a(a) \Rightarrow \forall b\, (P(b) \Rightarrow P_a(b)))$$

characterizes cyclic subgroups in A.

3. The formula

$$\begin{aligned}\mathrm{DCycl}(P(v)) := \mathrm{Gr}(P(v)) \wedge \forall a\, (P(a) \Rightarrow \exists P_1(v)\, \exists P_2(v)\, (P_1(a) \wedge \mathrm{Cycl}(P_1(v)) \\ \wedge \forall b\, \neg(P_1(b) \wedge P_2(b)) \wedge \forall b\, (P(b) \Rightarrow \exists b_1\, \exists b_2\, (P_1(b_1) \wedge P_2(b_2) \wedge b = b_1 + b_2)))\end{aligned}$$

characterizes those subgroups in A that are direct sums of cyclic subgroups.

4. For every $a, a_1, a_2 \in A$ the formula

$$\mathrm{Gr}_a(P_a(v)) := P_a(a) \wedge \mathrm{Gr}(P_a(v)) \wedge \forall P(v)\, (P(a) \wedge \mathrm{Gr}(P(v)) \Rightarrow \forall b\, (P_a(b) \Rightarrow P(b)))$$

defines in A the subgroup $\{b \in A \mid P_a(b)\}$ of all powers (exponents) of the element a; the formula

$$\begin{aligned}&(o(a_1) \leq o(a_2)) \\ &\quad := \exists P_1(v)\, \exists P_2(v)\, \exists P(v_1, v_2)\, (\mathrm{Gr}_{a_1}(P_1) \wedge \mathrm{Gr}_{a_2}(P_2) \wedge \forall b_1\, (P_1(b_1) \Rightarrow \exists b_2\, (P_2(b_2) \wedge P(b_1, b_2))) \\ &\quad \wedge \forall b_1\, \forall b_2\, \forall c_1\, \forall c_2\, (P_1(b_1) \wedge P_1(c_1) \wedge b_1 \neq c_1 \wedge P_2(b_2) \wedge P_2(c_2) \wedge P(b_1, b_2) \wedge P(c_1, c_2) \Rightarrow b_2 \neq c_2))\end{aligned}$$

holds if and only if the order of a_1 is not greater than the order of a_2; the formula

$$(o(a_1) = o(a_2)) := (o(a_1) \leq o(a_2)) \wedge (o(a_2) \leq o(a_1))$$

shows that the orders of a_1 and a_2 coincide; the formula

$$(o(a_1) < o(a_2)) := (o(a_1) \leq o(a_2)) \wedge \neg(o(a_2) \leq o(a_1))$$

shows that the order of a_1 is strictly smaller than the order of a_2.

5. For every $a \in A$ the formula

$$\mathrm{GOrd}_a(P(v)) := \mathrm{Gr}(P) \wedge \forall b\, (P(b) \Rightarrow o(b) \leq o(a))$$

holds for those subgroups that are bounded by the order of a, and only for them.

6. The formula

$$\begin{aligned}&\mathrm{Mult}_a(x, b) := \exists P(v)\, \exists P_{x,b}(v_1, v_2)\, (\mathrm{Cycl}(P) \wedge P(x) \wedge P(b) \wedge \forall b_1\, (P(b_1) \Rightarrow \exists b_2\, (P(b_2) \wedge P_{x,b}(b_1, b_2)) \\ &\quad \wedge (\forall b_1\, \forall b_2\, \forall b_3\, P(b_1) \wedge P_{x,b}(b_1, b_2) \wedge P_{x,b}(b_1, b_3) \Rightarrow b_2 = b_3) \\ &\quad \wedge (\forall b_1\, \forall b_2\, \forall b_3\, \forall c_1\, \forall c_2\, \forall c_3\, P(b_1) \wedge P(b_2) \wedge P(b_3) \\ &\quad \wedge b_3 = b_1 + b_2 \wedge c_3 = c_1 + c_2 \wedge P_{x,b}(b_1, c_1) \wedge P_{x,b}(b_2, c_2) \Rightarrow P_{x,b}(b_3, c_3)) \\ &\quad \wedge P_{x,b}(x, 0) \wedge \forall y\, (P(y) \wedge py = x \Rightarrow \neg P_{x,b}(y, 0)) \wedge \exists c\, (P(b, c) \wedge o(c) = o(a))))\end{aligned}$$

holds for elements x and b with the property $x = o(a) \cdot b$, and only for them.

7. The formula

$$\mathrm{Serv}(P(v)) := \mathrm{Gr}(P) \wedge \forall a\, \forall x\, (P(x) \Rightarrow\ \Rightarrow \exists b\, (\mathrm{Mult}_a(x, b) \Rightarrow \exists c\, (P(c) \wedge \mathrm{Mult}_a(x, c))))$$

holds for pure subgroups of the group A, and only for them.

8. The formula

$$\mathrm{FD}(P(v)) := \mathrm{Gr}(P) \wedge \forall a\, \exists b\, \exists x_1\, \exists x_2\, (P(x_1) \wedge P(x_2) \wedge a + x_1 = p(b + x_2))$$

holds for subgroups $G = \{x \mid P(x)\}$ such that A/G is a divisible subgroup, and only for them.

9. The formula

$$\mathrm{Base}(P(v)) := \mathrm{Gr}(P) \wedge \mathrm{DCycl}(P) \wedge \mathrm{Serv}(P) \wedge \mathrm{FD}(P)$$

defines basic subgroups in A.

It is clear that if we have some subgroup G' of the group G, then we similarly can write the formula $\mathrm{Base}_{G'}(P)$ which holds for basic subgroups of the group G', and only for them.

10. The formula

$$\mathrm{D}(P(v)) := \mathrm{Gr}(P) \wedge \forall a\, (P(a) \Rightarrow \exists b\, (P(b) \wedge a = pb))$$

defines divisible subgroups in A.

11. The sentence

$$\begin{aligned}\mathrm{Exept} := {} & \forall P\, (\mathrm{Gr}(P) \Rightarrow \neg(\mathrm{D}(P))) \\ & \wedge \forall P(v)\, (\mathrm{Base}(P) \Rightarrow \neg(\exists F(v_1, v_2)\, (\forall a\, (P(a) \Rightarrow \exists b\, (F(a,b)) \\ & \wedge \forall b\, \exists a\, (P(a) \wedge F(a,b)) \wedge \forall a\, \forall b\, (F(a,b) \Rightarrow P(a)) \\ & \wedge \forall a_1\, \forall a_2\, \forall b_1\, \forall b_2\, (a_1 \neq a_2 \wedge F(a_1,b_1) \wedge F(a_2,b_2) \Rightarrow b_1 \neq b_2) \\ & \wedge \forall b_1\, \forall b_2\, \forall a_1\, \forall a_2\, (b_1 \neq b_2 \wedge F(a_1,b_1) \wedge F(a_2,b_2) \Rightarrow a_1 \neq a_2)))))\end{aligned}$$

is true for reduced p-groups such that their basic subgroups have smaller power (and therefore are countable), and only for them. Thus if B_1 is a basic subgroup of the group A_1, B_2 is a basic subgroup of A_2, $\varkappa_1 = |B_1|$, and $\varkappa_2 = |B_2|$, then

$$\mathrm{Th}_2^{\varkappa_1}(A_1) = \mathrm{Th}_2^{\varkappa_2}(A_2)$$

implies that either the groups A_1 and A_2 are reduced, their basic subgroups are countable, and they themselves are uncountable, or this is not true for both of the groups A_1 and A_2.

In the first case $\varkappa_1 = \varkappa_2 = \omega$.

Theorem 3. *If Abelian groups A_1 and A_2 are reduced and their basic subgroups are countable, then* $\mathrm{Th}_2^{\omega}(A_1) = \mathrm{Th}_2^{\omega}(A_2)$ *implies elementary equivalence of the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$.

Proof. We know (see Theorem 10) that for a reduced p-group A the action of any endomorphism $\varphi \in \mathrm{End}(A)$ is completely defined by its action on a basic subgroup B. Furthermore, let $A' \subset A$ and let B be also a basic subgroup of A'. Then any $\varphi\colon A' \to A$ is also completely defined by its action on B. Indeed, if $\varphi_1, \varphi_2\colon A' \to A$ and $\varphi_1(b) = \varphi_2(b)$ for all $b \in B$, then for $\varphi := \varphi_1 - \varphi_2\colon A' \to A$ we have $\varphi(b) = 0$ for all $b \in B$. Hence φ induces a homomorphism $\tilde{\varphi}\colon A'/B \to A$. But the group A'/B is divisible and the group A is reduced, i.e., $\tilde{\varphi} = 0$. Consequently, $\varphi = 0$.

Note that for every element $a \in A$ there exists a countable subgroup $A' \subset A$ containing a and the group B as a basic subgroup.

Indeed, consider a quasibasis of the group A having the form

$$\{a_i, c_{j,n}\}_{i \in \omega,\ j \in \varkappa,\ n \in \omega},$$

where $\{a_i\}$ is a basis of B, $pc_{j,1} = 0$, $pc_{j,n+1} = c_{j,n} - b_{j,n}$, $b_{j,n} \in B$, $o(b_{j,n}) \leq p^n$, and $o(c_{j,n}) = p^n$.

As we remember, every element $a \in A$ can be written in the form

$$a = s_1 a_{i_1} + \cdots + s_m a_{i_m} + t_1 c_{j_1,n_1} + \cdots + t_r c_{j_r,n_r},$$

where s_i and t_j are integers, none of t_j is divisible by p and the indices $i_1, \ldots, i_m$, $j_1, \ldots, j_r$ are all distinct. Further, this form is unique in the sense that the members sa_i and $tc_{j,n}$ are uniquely defined.

Consider a decomposition of our element a and the subgroup in A generated by the group B and all $c_{k,n}$, where $n \in \omega$ and $k \in \{j_1, \ldots, j_r\}$. This group A' is countable, it contains a, and $B \subset A'$ is its basic subgroup.

Let now a predicate $B(v)$ satisfy in A the formula $\mathrm{Base}(B)$, i.e., $B(v)$ defines in A a basic subgroup $B = \{x \mid B(x)\}$.

A correspondence $P(v_1, v_2)$ is called a *homomorphism of the group B into the group A* (notation: $\mathrm{Hom}_B(P)$) if

$$\forall x\,(B(x) \Leftrightarrow \exists y\,(P(x,y))) \wedge \forall x\,\forall y_1\,\forall y_2\,(P(x,y_1) \wedge P(x,y_2) \Rightarrow y_1 = y_2) \\ \wedge \forall x_1\,\forall x_2\,\forall y_1\,\forall y_2\,(P(x_1,y_1) \wedge P(x_2,y_2) \Rightarrow P(x_1+x_2, y_2+y_2)).$$

It is clear that such a predicate $P(v_1, v_2)$ can be used in sentences from $\mathrm{Th}_2^{\omega}(A)$, because the group B is countable.

Consider some $B(v)$ such that the formula $\mathrm{Base}(B)$ is true, a predicate $\Phi(v_1, v_2)$ such that $\mathrm{Hom}_B(\Phi)$, and $a \in A$.

We shall write $b = \Phi(a)$ if

1. $B(a) \wedge \Phi(a,b)$ or
2. $\neg B(a) \wedge \forall G(v)\,(\mathrm{Gr}(G) \wedge G(a) \wedge \forall x\,(G(x) \Rightarrow B(x)) \wedge \mathrm{Base}_G(B) \Rightarrow \exists \varphi(v_1,v_2)\,(\forall x\,(G(x) \Leftrightarrow \exists y\,(\varphi(x,y)) \wedge \forall x\,\forall y_1\,\forall y_2\,(\varphi(x,y_1) \wedge \varphi(x,y_2) \Rightarrow y_1 = y_2) \wedge \forall x_1\,\forall x_2\,\forall y_1\,\forall y_2\,(\varphi(x_1,y_1) \wedge \varphi(x_2,y_2) \Rightarrow \varphi(x_1+x_2, y_1+y_2)) \wedge \forall x\,\forall y\,(\Phi(x,y) \Rightarrow \varphi(x,y)) \wedge \varphi(a,b))))$.

It is clear that for every $a \in A$ there exist no more than one $b \in A$ such that $b = \Phi(a)$, and if the homomorphism $\Phi\colon B \to A$ can be extended to an endomorphism $A \to A$, then it necessarily exists.

Now we shall consider $\Phi(v_1, v_2)$ such that

$$\mathrm{Endom}_B(\Phi) := \mathrm{Hom}_B(\Phi) \wedge \forall a\,\exists b\,(b = \Phi(a)) \\ \wedge \forall a_1\,\forall a_2\,\forall b_1\,\forall b_2\,(b_1 = \Phi(a_1) \wedge b_2 = \Phi(a_2) \Rightarrow b_1 + b_2 = \Phi(a_1 + a_2)).$$

In our case these $\Phi(v_1, v_2)$ define endomorphisms from $\mathrm{End}(A)$.

Let us show an algorithm of translation of formulas in this case.

A sentence φ is translated to the sentence

$$\tilde{\varphi} = \exists B(v)\,(\mathrm{Base}(B) \wedge \varphi'(B)),$$

where the formula φ' is obtained from the sentence φ in the following way:

1. the subformula $\forall x\,(\ldots)$ is translated to the subformula
$$\forall \Phi^x(v_1,v_2)\,(\mathrm{Endom}_B(\Phi^x) \Rightarrow \ldots);$$
2. the subformula $\exists x\,(\ldots)$ is translated to the subformula
$$\exists \Phi^x(v_1,v_2)\,(\mathrm{Endom}_B(\Phi^x) \wedge \ldots);$$
3. the subformula $x_1 = x_2$ is translated to the subformula
$$\forall y_1\,\forall y_2\,(\Phi^{x_1}(y_1,y_2) \Leftrightarrow \Phi^{x_2}(y_1,y_2));$$
4. the subformula $x_1 = x_2 + x_3$ is translated to the subformula
$$\forall y\,\forall z_1\,\forall z_2\,\forall z_3\,(\Phi^{x_2}(y,z_2) \wedge \Phi^{x_3}(y,z_3) \Rightarrow (\Phi^{x_1}(y,z_1) \Leftrightarrow z_1 = z_2 + z_3));$$
5. the subformula $x_1 = x_2 \cdot x_3$ is translated to the subformula
$$\forall y\,\forall z\,(\Phi^{x_1}(y,z) \Rightarrow \exists t\,(\Phi^{x_2}(y,t) \wedge z = \Phi^{x_3}(t)).$$

Now the proof is similar to the proof of the previous theorem. $\square$

4.4 Different Cases of the Problem

Following Theorem 11, we divide the class of all Abelian p-groups into the following three subclasses:

1. bounded p-groups;
2. the groups $D \oplus G$, where D is a nonzero divisible group and G is a bounded group;
3. groups with unbounded basic subgroups.

Now we shall show how to find sentences which distinguish groups from different subclasses.

If a group A is bounded, then there exists a natural number $n = p^k$ such that

$$\forall a\,(na = 0)$$

in the group A. This means that in the ring $\mathrm{End}(A)$ we also have

$$\forall x\,(nx = 0).$$

But if a group A is unbounded, then for every n this sentence is false. Therefore, we distinguish the case (1) from all other cases. Moreover, we now can find the supremum of orders of elements from A. We shall denote this sentence $\forall x\,(nx = 0)$ by φ_n.

Now consider the sentence

$$\begin{aligned}\psi_n := {} & \exists \rho_1\, \exists \rho_2\, (\rho_1\rho_2 = \rho_2\rho_1 = 0 \wedge \rho_1^2 = \rho_1 \wedge \rho_2^2 = \rho_2 \wedge \rho_1 + \rho_2 = 1\\ & \wedge \forall x\,(n \cdot \rho_2 x \rho_2 = 0) \wedge \forall \rho\, \forall \rho'\, (\rho^2 = \rho \wedge {\rho'}^2 = \rho' \wedge \rho\rho' = \rho'\rho = 0 \wedge \rho + \rho' = \rho_1\\ & \wedge \forall \tau_1\, \forall \tau_2\, (\tau_1^2 = \tau_1 \wedge \tau_2^2 = \tau_2 \wedge \tau_1\tau_2 = \tau_2\tau_1 = 0 \Rightarrow \tau_1 + \tau_2 \neq \rho) \Rightarrow \forall x\,(\rho x \rho = 0 \vee p(\rho x \rho) \neq 0))).\end{aligned}$$

We shall explain by words what this sentence means.

1. There exist orthogonal projections ρ_1 and ρ_2; their sum is 1 in the ring $\mathrm{End}(A)$. This means that $A = \rho_1 A \oplus \rho_2 A$, $\rho_1 \mathrm{End}(A) \rho_1 = \mathrm{End}(\rho_1 A)$, and $\rho_2 \mathrm{End}(A) \rho_2 = \mathrm{End}(\rho_2 A)$.

2. The condition $\forall x\,(n \cdot \rho_2 x \rho_2 = 0)$ means that in the ring $\mathrm{End}(\rho_2 A)$ all elements are bounded by a number $n = p^k$, i.e., the group $\rho_2 A$ is bounded.

3. The last part of the sentence ψ_n states that if in the ring $\mathrm{End}(\rho_1 A)$ we consider a primitive idempotent ρ (i.e., a projection onto an indecomposable direct summand $\rho A = \rho\rho_1 A$), then this direct summand does not have any endomorphisms of order p. Therefore in the group $\rho_1 A$ there are no cyclic direct summands, and so it is divisible.

Consequently, a ring $\mathrm{End}(A)$ satisfies the sentence ψ_n if and only if $A = D \oplus G$, where the group D is divisible and the group G is bounded by the number n.

Hence for any two groups A_1 and A_2 from different classes there exists a sentence which distinguishes the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$.

Thus we can now assume that if rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ are elementarily equivalent, then the groups A_1 and A_2 belong to one subclass, and if both of them belong to the first or the second subclass, then their reduced parts are bounded by the same number $n = p^k$, which is supposed to be fixed.

5 Bounded p-Groups

5.1 Separating Idempotents

As we have seen above (see Sec. 4.4), the property of $\rho \in \mathrm{End}(A)$ to be a decomposable idempotent which is a direct summand is a first order property. Let us denote the formula expressing this property by $\mathrm{Idem}^*(\rho)$, while the formula expressing the property of $\rho \in \mathrm{End}(A)$ to be simply an idempotent (not necessarily indecomposable) will be denoted by $\mathrm{Idem}(\rho)$.

We consider the group $A = \sum_{i=1}^{k} A_i$, where $A_i \cong \bigoplus_{\mu_i} \mathbb{Z}(p^i)$. Since the group A is infinite, we have that $\mu_l = \max_{i=1,\dots,k} \mu_i$ is infinite and coincides with $|A|$.

Consider for every $i = 1, \dots, k$ the formula

$$\mathrm{Idem}_i^*(\rho) = \mathrm{Idem}^*(\rho) \wedge p^{i-1}\rho \neq 0 \wedge p^i \cdot \rho = 0.$$

For every i this formula is true for projections on direct summands of the group A which are isomorphic to $\mathbb{Z}(p^i)$, and only for them.

Now consider the following formula:

$$\begin{aligned}
\mathrm{Comp}(\rho_1, \dots, \rho_k) = (\rho_1 + \dots + \rho_k = 1) \wedge \left(\bigwedge_{i \neq j} \rho_i \rho_j = \rho_j \rho_i = 0 \right) \\
\wedge \left(\bigwedge_{i=1}^{k} \rho_i^2 = \rho_i \right) \wedge \left(\bigwedge_{i=1}^{k} p^i \rho_i = 0 \right) \wedge \left(\bigwedge_{i=1}^{k} p^{i-1} \rho_i \neq 0 \right) \\
\wedge \left(\bigwedge_{i=1}^{k} \forall \rho \, (\mathrm{Idem}^*(\rho) \wedge \exists \rho' \, (\rho + \rho' = \rho_i \wedge \rho\rho' = \rho'\rho = 0 \wedge \mathrm{Idem}(\rho')) \Rightarrow \mathrm{Idem}_i^*(\rho)) \right).
\end{aligned}$$

We see that the group A (which corresponds to this formula) is decomposed into a direct sum $\rho_1 A \oplus \rho_2 A \oplus \dots \oplus \rho_k A = A$, and in every subgroup $\rho_i A$ all indecomposable direct summands have the order p_i. Therefore $\rho_1 A \oplus \dots \oplus \rho_k A$ is a decomposition of A, isomorphic to the decomposition $\sum_{i=1}^{k} A_i$.

Let us assume that the projections $\rho_1, \dots, \rho_k$ from the formula $\mathrm{Comp}(\dots)$ are fixed. To separate them from other idempotents we shall denote them by $\bar{\rho}_1, \dots, \bar{\rho}_k$. Having fixed idempotents $\bar{\rho}_1, \dots, \bar{\rho}_k$ of the ring $\mathrm{End}(A)$, we have also its fixed subrings

$$\bar{E}_i = \bar{\rho}_i \, \mathrm{End}(A) \bar{\rho}_i,$$

each of which is isomorphic to the ring

$$\mathrm{End}(\rho_i A) \cong \mathrm{End}(A_i).$$

Given an idempotent ρ satisfying the formula $\mathrm{Idem}_i^*(\rho)$ ($\mathrm{Idem}_i(\rho)$), the formula expressing for ρ the fact that it is a direct summand in the group $\bar{\rho}_i A$ (i.e., $\exists \rho' \, (\mathrm{Idem}(\rho') \wedge \rho\rho' = \rho'\rho = 0 \wedge \rho + \rho' = \bar{\rho}_i)$) will be written as $\overline{\mathrm{Idem}_i^*}(\rho)$ ($\overline{\mathrm{Idem}_i}(\rho)$). This formula means that the subgroup ρA is a direct summand in the group $\bar{\rho}_i A = A_i$.

The number l from the set $\{1, \dots, k\}$ which satisfies the sentence

$$\mathrm{Card}_l = \bigwedge_{i=1,\, i \neq l}^{k} \exists a \, \forall \rho \, (\overline{\mathrm{Idem}_i^*}(\rho) \Rightarrow \rho a \bar{\rho}_l \neq 0)$$

is the number of the group A_l with $|A_l| = |A| = \mu$, because this sentence means that there exists an endomorphism $a \in \mathrm{End}(A)$ mapping A_l to A in such a way that on every direct summand of A_i it is nonzero. This means that $|A_l| \geq |A_i|$, i.e., the power $|A_l|$ is maximal. Let us assume that also the number l is fixed.

The formula Card_l shows that we can write formulas that determine for every two projections ρ_1 and ρ_2 whether $|\rho_1 A| < |\rho_2 A|$, or $|\rho_1 A| > |\rho_2 A|$, or $|\rho_1 A| = |\rho_2 A|$ is true. Let us denote these formulas by $|\rho_1| < |\rho_2|$, $|\rho_1| > |\rho_2|$, and $|\rho_1| = |\rho_2|$, respectively.

The formula

$$\mathrm{Fin}(\rho) := \forall \rho_1 \, \forall \rho_2 \, (\mathrm{Idem}(\rho_1) \wedge \mathrm{Idem}(\rho_2) \wedge \mathrm{Idem}(\rho) \wedge \rho_1 = \rho + \rho_2 \wedge \rho\rho_2 = \rho_2\rho = 0 \Rightarrow |\rho| < |\rho_1|)$$

means that the group ρA is finitely generated. Respectively, the formula

$$\mathrm{Inf}(\rho) := \mathrm{Idem}(\rho) \wedge \neg \, \mathrm{Fin}(\rho)$$

holds for projections on infinitely generated groups.

The formula

$$\mathrm{Count}(\rho) := \mathrm{Inf}(\rho) \wedge \forall \rho_1 \, (\mathrm{Inf}(\rho_1) \Rightarrow |\rho| \le |\rho_1|)$$

is true for projections on countably generated groups and only for them.

Finally, we need the formula

$$\overline{\mathrm{Idem}_l^{\omega}}(\rho) = \overline{\mathrm{Idem}_l}(\rho) \wedge \mathrm{Count}(\rho),$$

which means that the group ρA is a countably generated direct summand of the group A_l.

5.2 Special Sets

At first we shall formulate what special sets we want to have. We must obtain two sets. One of them must contain μ_i independent indecomposable projections on direct summands of A_i, for every $i = 1, \ldots, k$, the other set must contain $\mu = \mu_l$ projections on independent countably generated direct summands of the group A_l.

By Theorem 1, we see that there exists a formula $\varphi(\bar{g}; f)$ satisfying the following condition. If $\{f_i\}_{i \in \mu}$ is a set of elements from $\mathrm{End}(A')$, then there exists a vector $\bar{g}$ such that the formula $\varphi(\bar{g}; f)$ is true in $\mathrm{End}(A')$ if and only if $f = f_i$ for some $i \in \mu$. We fix this formula φ.

Suppose that we have some fixed $i \in \{1, \ldots, k\}$. We have already shown that from the ring $\mathrm{End}(A)$ we can transfer to the ring $\mathrm{End}(A_i)$. Suppose that we argue in the ring $\mathrm{End}(A_i)$ (which satisfies the conditions of Theorem 1). In this ring let us consider the following formula:

$$\begin{aligned}
\tilde{\varphi}_i(\bar{g}) := {} & \forall f' \, (\varphi(\bar{g}, f') \Rightarrow \overline{\mathrm{Idem}_i^*}(f')) \\
& \wedge \forall f' \, (\overline{\mathrm{Idem}_i}(f') \wedge \forall f_1 \, (\varphi(\bar{g}, f_1) \Rightarrow \exists f_2 \, (\overline{\mathrm{Idem}_i}(f_2) \wedge f_1 f_2 = f_2 f_1 = 0 \wedge f_1 + f_2 = f')) \Rightarrow |f'| = |\rho_i|) \\
& \wedge \forall f' \, (\varphi(\bar{g}, f') \Rightarrow (\exists f \, (\overline{\mathrm{Idem}_i}(f) \wedge \forall f_1 \, (\varphi(\bar{g}, f_1) \wedge f_1 \ne f' \Rightarrow f_1 f = f f_1 = f_1) \wedge f f' = f' f = 0))).
\end{aligned}$$

The part $\forall f' \, (\varphi(\bar{g}, f') \Rightarrow \overline{\mathrm{Idem}_i^*}(f'))$ means that the vector $\bar{g}$ is such that the formula $\varphi(\bar{g}, f)$ is true only for projections f on indecomposable direct summands of the group A_i.

The part $\forall f' \, (\overline{\mathrm{Idem}_i}(f') \wedge \forall f_1 \, (\varphi(\bar{g}, f_1) \Rightarrow \exists f_2 \, (\overline{\mathrm{Idem}_i}(f_2) \wedge f_1 f_2 = f_2 f_1 = 0 \wedge f_1 + f_2 = f')) \Rightarrow |f'| = |\rho_i|)$ means that those subgroups of the group A_i that contain all summands fA satisfying $\varphi(\bar{g}, f)$ have the same power as A_i, i.e., this part means that the power of the set of these f is equal to μ.

The last part of the formula means that for every f' satisfying $\varphi(\bar{g}, f')$, the group generated by all other f satisfying $\varphi(\bar{g}, f)$ does not intersect with f', i.e., the set of all f satisfying $\varphi(\bar{g}, f)$ is independent.

This set will be denoted by $\mathbf{F}_i$. It consists of μ_i independent projections on indecomposable direct summands of the group A_i. Naturally, this set can be obtained for every vector $\bar{g}_i$ satisfying the formula $\tilde{\varphi}_i(\bar{g}_i)$, therefore we have to write not $\mathbf{F}_i$, but $\mathbf{F}_i(\bar{g}_i)$, and we shall do so in what follows. But in the cases where parameters are not so important we shall omit them.

The union of all $\mathbf{F}_i$ for $i = 1, \ldots, k$ will be denoted by $\mathbf{F}$. The set $\mathbf{F}$ depends on the parameter $\bar{g} = (\bar{g}_1, \ldots, \bar{g}_k)$.

Now we need to obtain a set $\mathbf{F}'$ consisting of $\mu_l = \mu$ independent projections on countably generated direct summands of A_l. It will be done similarly to the previous case, we only need to change in the formula $\tilde{\varphi}_l(\bar{g}')$ the following parts: $\overline{\mathrm{Idem}_l^*}(f)$ to $\overline{\mathrm{Idem}_l^{\omega}}(f)$; besides, we shall consider vectors $\bar{g}'$ such that

1. $\forall f \in \mathbf{F}_l \, (\exists f' \, (\varphi(\bar{g}', f') \wedge f f' = f))$, i.e., for every cyclic direct summand fA (where $f \in \mathbf{F}_l$) of A_l there exists a countably generated summand $f'A$ of A_l such that $\varphi(\bar{g}', f')$ and $fA \subset f'A$;

2. (we shall write it by words, because we do not want to write complicated formulas) every direct summand in A_l which contains all fA for all projections f such that $\varphi(\bar{g}_l, f)$ contains all $f'A$ such that $\varphi(\bar{g}', f')$.

Denote the corresponding formula by $\tilde{\tilde{\varphi}}_l(\bar{g}')$ and the obtained set of projections by $\mathbf{F}' = \mathbf{F}'(\bar{g}')$.

5.3 Interpretation of the Group A for Every Element $\mathbf{F}'$

By interpretation of the group A for every element from $\mathbf{F}'$ we understand the following. We have μ independent direct summands $\mathcal{F}_i = f_i A$ $(i \in \mu)$ each of which is a direct sum of a countable set of cyclic groups of order p^l. Every endomorphism of the group A acts independently on every summand $\mathcal{F}_i$, hence if for every endomorphism $\varphi \in \mathrm{End}(A)$ we can map every element of $\mathcal{F}_i$ to some element of A, then we shall be able to map every endomorphism $\varphi \in \mathrm{End}\, A$ to a set of μ elements of A. This is what we need below to obtain the second order theory of the group A. So in this section, we shall concentrate on a bijective correspondence between some homomorphisms from the group $\mathcal{F}_i$ into the group A, and elements of the group A, and introduce on the set of homomorphisms an operation $\oplus$ that under this bijection corresponds to the addition of the group A.

Let us fix some projection $g \in \mathbf{F}'$. Consider the set End_g of all those homomorphisms $h\colon gA \to A$ that satisfy the following conditions:

1. $\forall f \in \mathbf{F}_l\, (fg = f \Rightarrow (hf = 0 \vee \exists f' \in \mathbf{F}\, (hf = f'hf \neq 0)))$; this means that for every projection f from our special set $\mathbf{F}_l$, if the projection maps A to the indecomposable direct summand fA of the module gA, then either $h(fA) = 0$ or $h(fA) \subset f'A$ for some projection $f' \in \mathbf{F}$;

2. $\exists f\, (\mathrm{Fin}(f) \wedge \mathrm{Idem}(f) \wedge fh = h)$; this means that the image of the subgroup gA under the endomorphism h is finitely generated;

3. $\bigwedge\limits_{i=1}^{k} \forall f \in \mathbf{F}_i\, \neg\Big(\exists f_1 \ldots \exists f_{p^i} \Big(\bigwedge\limits_{q\neq s} f_q \neq f_s \wedge f_1, \ldots, f_{p^i} \in F_l \wedge f_1 g = g_1 \wedge \ldots \wedge f_{p^i} g = f_{p^i} \wedge hf_1 = fhf_1 \neq 0 \wedge \ldots \wedge hf_{p^i} = fhf_{p^i} \neq 0\Big)\Big)$; this means that for every $i = 1, \ldots, k$ the inverse image of each $fA \subset A_i$, where $f \in \mathbf{F}_i$, can not contain more than $p^i - 1$ different elements $f_m A$, where $f_m A \subset gA$ and $f_m \in \mathbf{F}_l$.

Two elements h_1 and h_2 from the set End_g are said to be equivalent $(h_1 \sim h_2)$ if they satisfy the following formula:

$$\begin{aligned}&\exists f_1\, \exists f_2\, ((gf_1g) \cdot (gf_2g) = (gf_2g) \cdot (gf_1g) = g \\ &\qquad \wedge \forall f \in \mathbf{F}_l\, (fg = f \Rightarrow \forall f' \in \mathbf{F}\, (h_1 f = f'h_1 f \neq 0 \Leftrightarrow (gf_1gh_2)f = f'(gf_1gh_2)f \neq 0)).\end{aligned}$$

This means that there exists an automorphism gf_1g of the group gA which maps h_2 to an endomorphism $(gf_1g \cdot h_2)$ such that for every $\rho \in F_l$, where $\mathrm{Im}\,\rho = gA$, both endomorphisms h_1 and gf_1gh_2 map this subgroup either to zero or to the same $f'A$ $(f' \in \mathbf{F})$. The obtained set $\mathrm{End}_g/\!\sim$ will be denoted by $\widetilde{\mathrm{End}}_g$. We can interpret elements of this set as finite sets of projections from $\mathbf{F}$ with the condition that every projection from $\mathbf{F}_i$ can belong to this set at most $p^i - 1$ times. Respectively, every element of the set $\widetilde{\mathrm{End}}_g$ can be interpreted as a set of pairs, where the first element in a pair is a projection f from $\mathbf{F}$ and the second element is an integer from 0 to $p^i - 1$, where i is such that $f \in \mathbf{F}_i$, and almost all (all except for a finite number) second components of the pairs are equal to 0. Now we can construct a bijective mapping between the set $\widetilde{\mathrm{End}}_g$ and the group A, where the image of the described set $\{\langle f_j, l_j\rangle \mid j \in J\}$ is the element $\sum\limits_{j\in J} l_j \xi_j = a \in A$, where ξ_j is some fixed generator of the cyclic group $f_j A$.

Now we only need to introduce addition on the set $\widetilde{\mathrm{End}}_g$ to make the obtained bijective mapping an isomorphism of Abelian groups.

We shall introduce addition by the formula $(h_1, h_2, h_3 \in \widetilde{\mathrm{End}}_g)$

$$\begin{aligned}(h_3 = h_1 \oplus h_2) := \bigwedge_{i=1}^{k} \forall f \in \mathbf{F}_i \Bigg(\bigwedge_{j=0}^{p^i-1} \exists g_1 \ldots \exists g_j \in \mathbf{F}_l \bigwedge_{q\neq s} (g_q \neq g_s \wedge g_q g = g_q \\ \wedge h_3 g_q = fh_3 g_q \neq 0) \wedge \neg\Big(\exists g_1 \ldots \exists g_{j+1} \in \mathbf{F}_l \bigwedge_{q \neq s} (g_q \neq g_s \wedge g_q g = g_q \wedge h_3 g_q = fh_3 g_q \neq 0)\Big)\end{aligned}$$

$$\Rightarrow \Bigg(\bigvee_{m=0}^{j} \exists g_1 \ldots \exists g_m \in \mathbf{F}_l \bigwedge_{q \neq s} (g_q \neq g_s \wedge g_q g = g_q \wedge h_1 g_q = f h_1 g_q \neq 0)$$

$$\wedge \neg \Bigg(\exists g_1 \ldots \exists g_{m+1} \in \mathbf{F}_l \bigwedge_{q \neq s} (g_q \neq g_s \wedge g_q g = g_q \wedge h_1 g_q = f h_1 g_q \neq 0) \Bigg)$$

$$\wedge \exists g_1 \ldots \exists g_{\gamma(j,m)} \in \mathbf{F}_l \bigwedge_{q \neq s} (g_q \neq g_s \wedge g_q g = g_q \wedge h_2 g_q = f h_2 g_q \neq 0)$$

$$\wedge \neg \Bigg(\exists g_1 \ldots \exists g_{\gamma(j,m)+1} \in \mathbf{F}_l \bigwedge_{q \neq s} (g_q \neq g_s \wedge g_q g = g_q \wedge h_2 g_q = f h_2 g_q \neq 0) \Bigg) \Bigg) \Bigg),$$

where $\gamma(j,m) = j - m$ if $j \geq m$, and $\gamma(j,m) = p^i + j - m$ if $j < m$.

Now we see that for every $g \in \mathbf{F}'$ we have a definable set $\widetilde{\mathrm{End}}_g$ with the addition operation $\oplus$, which is isomorphic to the group A.

5.4 Proof of the First Case in the Theorem

Proposition 1. *For any two infinite Abelian p-groups A_1 and A_2 bounded by the number p^k, elementary equivalence of the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *implies equivalence of the groups A_1 and A_2 in the language $\mathcal{L}_2$.*

Proof. For every $\tilde{g} \in \mathbf{F}'$ by $\mathrm{Resp}_{\tilde{g}}(h)$ we shall denote the following formula:

$$\mathrm{Resp}_{\tilde{g}}(h) := \forall g \in \mathbf{F}' \, \exists h' \, ((\tilde{g} h g)(g h' \tilde{g}) = \tilde{g} \wedge (g h' \tilde{g})(\tilde{g} h g) = g).$$

This formula means that an endomorphism h isomorphically maps every summand gA (where $g \in \mathbf{F}'$) to the summand $\tilde{g}A$.

As above, let us consider an arbitrary sentence φ in the second order group language and show an algorithm translating this sentence ψ to a sentence $\tilde{\psi}$ of the first order ring language so that $\mathrm{End}(A) \vDash \tilde{\psi}$ if and only if $A \vDash \varphi$.

Let us translate the sentence ψ to the sentence

$$\exists \bar{g}_1 \ldots \exists \bar{g}_k \, (\tilde{\varphi}_1(\bar{g}_1) \wedge \ldots \wedge \tilde{\varphi}_k(\bar{g}_k) \wedge \exists \bar{g}' \, (\tilde{\tilde{\varphi}}_l(\bar{g}', \bar{g}_l) \wedge \exists \tilde{g} \in \mathbf{F}'(\bar{g}') \, \exists h \, (\mathrm{Resp}_{\tilde{g}}(h) \wedge \psi'(\bar{g}_1, \ldots, \bar{g}_k, \bar{g}', \tilde{g}, h)))),$$

where the formula $\psi'(\ldots)$ is obtained from the sentence ψ with the help of the following translations of subformulas of ψ:

1. the subformula $\forall x$ is translated to the subformula $\forall x \in \widetilde{\mathrm{End}}_{\tilde{g}}$;
2. the subformula $\exists x$ is translated to the subformula $\exists x \in \widetilde{\mathrm{End}}_{\tilde{g}}$;
3. the subformula $\forall P_m(v_1, \ldots, v_m)\,(\ldots)$ is translated to the subformula
$$\forall f_1^P \ldots \forall f_m^P \left(\forall g \in \mathbf{F}'(\bar{g}') \left(\bigwedge_{i=1}^{m} (f_i^P g \in \mathrm{End}_g) \right) \Rightarrow \ldots \right);$$
4. the subformula $\exists P_m(v_1, \ldots, v_m)\,(\ldots)$ is translated to the subformula
$$\exists f_1^P \ldots \exists f_m^P \left(\forall g \in \mathbf{F}'(\bar{g}') \left(\bigwedge_{i=1}^{m} (f_i^P g \in \mathrm{End}_g) \right) \wedge \ldots \right);$$
5. the subformula $x_1 = x_2$ is translated to the subformula $x_1 \sim x_2$;

6. the subformula $x_1 = x_2 + x_3$ is translated to the subformula $x_1 \sim x_2 \oplus x_3$;
7. the subformula $P_m(x_1, \dots, x_m)$ is translated to the subformula

$$\exists g \in \mathbf{F}'(\bar{g}') \left(\bigwedge_{i=1}^{m} f_i^P g = x_i hg \right).$$

We can explain by words what these translations mean. According to existence of the set $\mathbf{F}'$, we have μ groups $\widetilde{\mathrm{End}}_g$ for $g \in \mathbf{F}'$, each of which is isomorphic to the group A. We fix one chosen element $\tilde{g} \in \mathbf{F}'$, and therefore we fix one group $\widetilde{\mathrm{End}}_{\tilde{g}}$, isomorphic to A. Naturally, all subformulas $\forall x$, $\exists x$, $x_1 = x_2$, $x_1 = x_2 + x_3$ (of first order logic) will be translated to the corresponding subformulas for the group $\widetilde{\mathrm{End}}_{\tilde{g}}$. Now we need to interpret an arbitrary relation $P_m(v_1, \dots, v_m)$ on A in the ring $\mathrm{End}(A)$. Such a relation is some subset in A^m, i.e., a set of ordered m-tuples of elements from A. There are at most μ such m-tuples, therefore the set $P_m(v_1, \dots, v_m)$ can be considered as a set of μ m-tuples of elements from A (some of them can coincide). We consider m endomorphisms $f_1^P, \dots, f_m^P \in \mathrm{End}(A)$ such that the restriction of each of them on any gA (where $g \in \mathbf{F}'$) is an element of $\widetilde{\mathrm{End}}_g$. Thus for every $g \in \mathbf{F}'$ the restriction of the endomorphisms $f_1^P, \dots, f_m^P$ on gA is an m-tuple of elements of the group $\widetilde{\mathrm{End}}_{\tilde{g}}$ ($\cong A$), where an isomorphism between $\widetilde{\mathrm{End}}_{\tilde{g}}$ and $\widetilde{\mathrm{End}}_g$ is given by the fixed mapping h which isomorphically maps every module gA to $\tilde{g}A$.

So we can see that the sentence ψ is true in A if and only if the sentence $\tilde{\psi}$ is true in the ring $\mathrm{End}(A)$. Therefore, as in the previous section, we have the proof. $\square$

6 Direct Sums of Divisible and Bounded p-Groups

6.1 Finitely Generated Groups

Every infinite finitely generated Abelian p-group A has the form $D \oplus G$, where D is a divisible finitely generated group and G is a finite group. There is no need to prove the following proposition.

Proposition 1. *If Abelian p-groups A_1 and A_2 are finitely generated, then elementary equivalence of their endomorphism rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *implies that the groups A_1 and A_2 are isomorphic.*

6.2 Infinitely Generated Divisible Groups

As in Sec. 5.1, the formula $\mathrm{Idem}^*(\rho)$ will denote the property of an endomorphism ρ to be an indecomposable idempotent, while $\mathrm{Idem}(\rho) := (\rho^2 = \rho)$. If in a divisible group D we have $\mathrm{Idem}^*(\rho)$, then $\rho A \cong \mathbb{Z}(p^\infty)$.

Note that, despite the fact that in Sec. 3 we considered direct sums of cyclic groups of the same order, the Shelah theorem remains true also for divisible groups, because a divisible group is a union of the groups

$$\bigoplus_{\mu} \mathbb{Z}(p) \subset \bigoplus_{\mu} \mathbb{Z}(p^2) \subset \dots \subset \bigoplus_{\mu} \mathbb{Z}(p^n) \subset \dots$$

(proof of an even more general case is given later in Sec. 7). Therefore, similarly to Sec. 5.2, we have a definable set $\mathbf{F} = \mathbf{F}(\bar{g})$ consisting of μ indecomposable projections on linearly independent direct summands of D, and also a definable set $\mathbf{F}' = \mathbf{F}'(\bar{g}')$ consisting of μ projections on linearly independent countably generated direct summands of D.

Let us fix some element $g \in \mathbf{F}'$ and construct (as in Sec. 5) an interpretation of the group D for this set.

Namely, let us consider the set End_g of all homomorphisms $h \colon gA \to A$ satisfying the following conditions:

1. $\forall f \in \mathbf{F}\, (fg = f \Rightarrow (hf = 0 \vee \exists \tilde{f} \in \mathbf{F}\, (\tilde{h}f = \tilde{f}hf \neq 0)))$; this means that for every projection f from $\mathbf{F}$ such that $fA \subset gA$, we have either $h(fA) = 0$ or $h(fA) \subset \tilde{f}A$ for some $\tilde{f} \in \mathbf{F}$;

2. $\exists f\,(\mathrm{Fin}(f) \wedge \mathrm{Idem}(f) \wedge fh = h)$; this means that the image of gA on the endomorphism h is finitely generated;

3. $\forall f \in \mathbf{F}\,(\exists f' \in \mathbf{F}\,(f'g = g' \wedge hf' = fhf' \neq 0) \Rightarrow \exists \tilde{f} \in \mathbf{F}\,(\tilde{f}g = \tilde{f} \wedge h\tilde{f} = fh\tilde{f} \neq 0) \wedge \forall f' \in \mathbf{F}$ $(f'g = f' \wedge hf' = fhf' \neq 0 \Rightarrow \exists \alpha\,(\alpha hf' = h\tilde{f})))$; this means that for every element $f \in \mathbf{F}$ either the inverse image of fA is empty or it contains an element $\tilde{f}A \subset gA$ (where $\tilde{f} \in \mathbf{F}$) such that the kernel $\tilde{f}A$ on the mapping h has the maximal order among all kernels $f'A$ such that $f' \in \mathbf{F}$ and $f'A \subset gA$.

Before the last condition we shall introduce some new notation. Let h be some endomorphism and $f_1, f_2 \in \mathbf{F}$. We shall write $f_1 \sim_h f_2$ if and only if the formula

$$\exists \alpha\,(\alpha^2 = 1 \wedge \alpha f_1 = f_2 \alpha \wedge \alpha f_2 = f_1 \alpha \wedge hf_1 = h\alpha f_1 \wedge hf_2 = h\alpha f_2)$$

is true. This formula means that the images of the groups f_1A and f_2A on the endomorphism h coincide and the kernels of the groups f_1A and f_2A on this endomorphism are isomorphic.

Now for an endomorphism h and projections $f_1, f_2 \in \mathbf{F}$ we shall introduce the formula

$$\exists \alpha\,(\alpha^2 = 1 \wedge \alpha f_1 = f_2 \alpha \wedge \alpha f_2 = f_1 \alpha \wedge hf_1 = ph\alpha f_1).$$

This formula states that the images of the groups f_1A and f_2A on the endomorphism h coincide and the kernel of f_1A is p times greater than the kernel of f_2A. This formula will be denoted by $f_1 \sim_h f_2 + 1$.

Now we shall introduce the last condition:

4. $\neg\Big(\exists f_1, \ldots, f_p \in \mathbf{F}\,\Big(\Big(\bigwedge\limits_{i \neq j} f_i \neq f_j\Big) \wedge f_1 g = f_1 \wedge \ldots \wedge f_p g = f_p \wedge hf_1 = fhf_1 \neq 0 \wedge \ldots \wedge hf_p = fhf_p \neq 0 \wedge \Big(\bigwedge\limits_{i \neq j} f_i \sim f_j\Big)\Big)\Big)$; this means that there exist at most $p-1$ such projections from $\mathbf{F}$ onto subgroups of gA that their kernels are isomorphic.

Two elements h_1 and h_2 of the set End_g are said to be equivalent ($h_1 \sim h_2$) if the following formula holds:

$$\exists f_1\, \exists f_2\,((gf_1g)\cdot(gf_2g) = (gf_2g)\cdot(gf_1g) = g \wedge \forall f \in \mathbf{F}\,(fg = f \Rightarrow h_1 f = (gf_1g)h_2 f \wedge h_2 f = (gf_2g)h_1 f)).$$

This means that there exist mutually inverse automorphisms gf_1g and gf_2g of the group gA which map h_2 and h_1 to automorphisms $(gf_1g)h_2$ and $(gf_2g)h_1$ such that we have $h_2 = (gf_2g)h_1$ and $h_1 = (gf_1g)h_2$ on gA for every projection from $\mathbf{F}$ projecting on the subgroup of gA.

The obtained set $\mathrm{End}_g /{\sim}$ will be denoted by $\widetilde{\mathrm{End}}_g$.

Suppose that we have two quasicyclic groups C and C', one of which has generators $c_1, \ldots, c_n, \ldots$ ($pc_1 = 0$, $pc_{n+1} = c_n$) and the other one has $c'_1, \ldots, c'_n, \ldots$ ($pc'_1 = 0$, $pc'_{n+1} = c'_n$). Consider the set of homomorphisms $\mathrm{Hom}(C, C')$. Two homomorphisms $\alpha_1, \alpha_2 \in \mathrm{Hom}(C, C')$ correspond to each other under some automorphism of the group C if and only if their kernels are isomorphic, i.e., have the same order. Thus, all homomorphisms from $\mathrm{Hom}(C, C')$ are divided into a countable number of classes, and every class uniquely corresponds to a nonnegative integer i such that $|\mathrm{Ker}\,\alpha| = p^i$.

Consequently, every class $h \in \widetilde{\mathrm{End}}_g$ can be mapped to a finite set of finite sequences

$$\langle f, m(f), l_1(f), \ldots, l_{m(f)}(f)\rangle,$$

where $f \in \mathbf{F}$, $m(f) \in \mathbb{N}$, and $l_i(f) = 0, \ldots, p-1$. It is clear that endomorphisms from the same equivalence class are mapped to the same sets, and endomorphisms from different classes are mapped to different sets. Moreover, it is clear that every finite set of sequences is mapped to some class of endomorphisms. Now every such set of sequences will be mapped to an element

$$\sum_{f \in \mathbf{F}} l_1(f)c_1(f) + \cdots + l_m(f)c_m(f),$$

where $c_1(f), \ldots, c_n(f), \ldots$ are some fixed generators of fA.

Therefore we have obtained a bijection between the set $\widetilde{\mathrm{End}}_g$ and the group $A \cong \bigoplus_{\mu} \mathbb{Z}(p^\infty)$.

Now let us introduce on the set $\widetilde{\mathrm{End}}_g$ an addition ($h_3 = h_1 \oplus h_2$) in such a way that this bijection becomes an isomorphism between Abelian groups.

Let $h_1, h_2, h_3 \in \widetilde{\mathrm{End}}_g$.

$$
\begin{aligned}
(h_3 = h_1 \oplus h_2) := {} & \forall f \in \mathbf{F} \Big(\forall f' \in \mathbf{F} \Big(f'g = f' \wedge h_1 f' = f h_1 f' \neq 0 \\
& \Rightarrow \bigwedge_{i=0}^{p-1} \Big(\exists f_1 \ldots \exists f_i \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_1} f' \wedge f_q g = f_q \wedge h_1 f_q = f h_1 f_q \neq 0) \\
& \wedge \neg \Big(\exists f_1, \ldots, \exists f_{i+1} \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_1} f' \wedge f_q g = f_q \wedge h_1 f_q = f h_1 f_q \neq 0) \Big) \\
& \Rightarrow \bigvee_{j=0}^{p-1} \Big(\exists f_1 \ldots \exists f_j \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_2} f' \wedge f_q g = f_q \wedge h_2 f_q = f h_2 f_q \neq 0) \\
& \wedge \neg \Big(\exists f_1 \ldots \exists f_{j+1} \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_2} f' \wedge f_q g = f_q \wedge h_2 f_q = f h_2 f_q \neq 0) \Big) \\
& \wedge \Big(\Big(\exists f_1 \ldots \exists f_{(i+j) \bmod p} \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_3} f' \wedge f_q g = f_q \wedge h_3 f_q = f h_3 f_q \neq 0) \\
& \wedge \neg \Big(\exists f_1 \ldots \exists f_{(i+j)\, mod\, p+1} \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_3} f' \wedge f_q g = f_q \wedge h_3 f_q = f h_3 f_q \neq 0) \Big) \\
& \wedge \neg \Big(\exists f_1 \ldots f_p \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_3} f' + 1 \\
& \wedge f_q g = f_q \wedge (h_1 f_q = f h_1 f_q \neq 0 \vee h_2 f_q = f h_2 f_q \neq 0)) \Big) \Big) \\
& \vee \Big(\exists f_1 \ldots \exists f_{(i+j) \bmod p+1} \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_3} f' \wedge f_q g = f_q \wedge h_3 f_q = f h_3 f_q \neq 0) \\
& \wedge \neg \Big(\exists f_1 \ldots \exists f_{(i+j) \bmod p+2} \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_3} f' \wedge f_q g = f_q \wedge h_3 f_q = f h_3 f_q \neq 0) \Big) \\
& \wedge \exists f_1 \ldots f_p \in \mathbf{F} \bigwedge_{q \neq s} (f_q \neq f_s \wedge f_q \sim_{h_3} f' + 1 \wedge f_q g = f_q \\
& \wedge (h f_q = f h_1 f_q \neq 0 \vee h_2 f_q = f h_2 f_q \neq 0)) \Big) \Big) \Big) \Big) \Big) \Big) \Big).
\end{aligned}
$$

Hence for every $g \in \mathbf{F}'$ there is a definable set $\widetilde{\mathrm{End}}_g$ with the addition operation $\oplus$, which is isomorphic to A.

Proposition 2. *For two infinitely generated divisible p-groups A_1 and A_2, elementary equivalence of the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *implies equivalence of the groups A_1 and A_2 in the language $\mathcal{L}_2$.*

Proof. Since we have obtained an interpretation of the group A for every $g \in \mathbf{F}'$, the proof of this proposition is completely similar to the proof of Proposition 1. $\square$

6.3 Direct Sums of Divisible p-Groups and Bounded p-Groups of Not Greater Power

In this section, we consider the groups of the form $D \oplus G$, where D is an infinitely generated divisible group, G is a group bounded by the number p^k, and $|G| \leq |D|$. This case is practically the union of the previous two cases.

Namely, let us have idempotents ρ_D and ρ_G from the formula ψ_{p^k} of Sec. 4.4, i.e., idempotents which are projections on divisible and bounded parts of A, respectively, and also idempotents $\rho_1, \dots, \rho_k$, where $\rho_1 + \dots + \rho_k = \rho_G$, which are projections on direct summands of the form $\bigoplus_{\mu_1} \mathbb{Z}(p), \dots, \bigoplus_{\mu_k} \mathbb{Z}(p^k)$, respectively. Let $|A| = |D| = \mu$. As before, we have the following definable sets:

1. $\mathbf{F} = \mathbf{F}(\bar{g})$ is a set of μ indecomposable projections on linearly independent direct summands of the group D;
2. the set $\mathbf{F}' = \mathbf{F}'(\bar{g}')$ consists of μ projections on linearly independent countably generated direct summands of D;
3. for every $i = 1, \dots, k$ the set $\mathbf{F}_i = \mathbf{F}_i(\bar{g}_i)$ consists of μ_i projections on independent indecomposable direct summands of $\rho_i A$;
4. an endomorphism $\varphi \in \mathrm{End}(A)$ satisfying the following formula:

$$\begin{aligned}
&\forall f \in \mathbf{F}(\bar{g})\, (\varphi f \in \mathbf{F}) \wedge \bigwedge_{i=1}^{k} (\rho_D \varphi \rho_i = \varphi \rho_i \wedge \forall f_i \in \mathbf{F}_i(\bar{g}_i)\, \exists f \in \mathbf{F}(\bar{g})\, (\varphi f_i = f \varphi f_i \neq 0)) \\
&\quad \wedge \forall f_1, f_2 \in \mathbf{F}(\bar{g})\, (f_1 \neq f_2 \Rightarrow \forall f_1', f_2' \in \mathbf{F}(\bar{g})\, (\varphi f_1 = f_1' \varphi f_1 \neq 0 \wedge \varphi f_1 = f_2' \varphi f_2 \neq 0 \Rightarrow f_1' \neq f_2') \\
&\quad \wedge \bigwedge_{i=1}^{k} \forall f \in \mathbf{F}(\bar{g})\, \forall f_i \in \mathbf{F}_i(\bar{g}_i)\, \forall f' \in F(\bar{g})\, (f' = \varphi f \Rightarrow f' \varphi f_i = 0) \\
&\quad \wedge \bigwedge_{i,j=1}^{k} \forall f_1, f_2 \in \mathbf{F}(\bar{g})\, \forall f_i \in \mathbf{F}_i(\bar{g}_i)\, \forall f_j \in \mathbf{F}_j(\bar{g}_j) \\
&\quad (f_i \neq f_j \wedge \varphi f_i = f_1 \varphi f_i \neq 0 \wedge \varphi f_j = f_2 \varphi f_j \neq 0 \Rightarrow f_1 \neq f_2)).
\end{aligned}$$

We see that such an endomorphism φ embeds the set

$$\mathbf{F}(\bar{g}) \cup \mathbf{F}_1(\bar{g}_1) \cup \dots \cup \mathbf{F}_k(\bar{g}_k)$$

into the set $\mathbf{F}(\bar{g})$. Therefore for a given φ we can consider the sets

$$\begin{aligned}
\mathbf{F}^D &= \mathbf{F}^D(\bar{g}, \varphi) = \{ f \in \mathbf{F}(\bar{g}) \mid \exists f' \in \mathbf{F}(\bar{g})\, (f \varphi f' = f \varphi = \varphi f' \neq 0) \}, \\
\mathbf{F}_1^D &= \mathbf{F}_1^D(\bar{g}_1, \varphi) = \{ f \in \mathbf{F}(\bar{g}) \mid \exists f' \in \mathbf{F}_1(\bar{g}_1)\, (f \varphi f' = f \varphi = \varphi f' \neq 0) \}, \\
&\dots \\
\mathbf{F}_k^D &= \mathbf{F}_k^D(\bar{g}_k, \varphi) = \{ f \in \mathbf{F}(\bar{g}) \mid \exists f' \in \mathbf{F}_k(\bar{g}_k)\, (f \varphi f' = f \varphi = \varphi f' \neq 0) \}.
\end{aligned}$$

The sets $\mathbf{F}^D, \mathbf{F}_1^D, \dots, \mathbf{F}_k^D$ consist of $\mu, \mu_1, \dots, \mu_k$ projections on indecomposable linearly independent direct summands of the group D, respectively. We shall write them in formulas, sometimes omitting parameters, but meaning that they depend on the parameters $\bar{g}, \bar{g}_1, \dots, \bar{g}_k, \varphi$.

Let us fix some element $g \in \mathbf{F}'$ and construct an interpretation of the group $A = D \oplus G$ for this set.

Namely, let us consider the set End_g of all homomorphisms $h \colon gA \to A$ satisfying the following conditions:

1. $\forall f \in \mathbf{F} \Big(fg = f \Rightarrow \Big(hf = 0 \vee \exists \tilde{f} \in \mathbf{F}\, (hf = \tilde{f} hf \neq 0) \vee \bigwedge_{i=1}^{k} \exists \tilde{f} \in \mathbf{F}_i^D\, (\tilde{f} h = \tilde{f} hf = hf \neq 0) \Big) \Big)$; this means that for every projection f from $\mathbf{F}$, if $fA \subset gA$, then we have either $h(fA) = 0$, or $h(fA) \subset \tilde{f} A$ for some $\tilde{f} \in \mathbf{F}^D$, or $h(fA) = \tilde{f} A$ for some $\tilde{f} \in \mathbf{F}_i^D$;

2. $\exists f\,(\mathrm{Fin}(f) \wedge \mathrm{Idem}(f) \wedge fh = h)$; this means that the image of gA under h is finitely generated;

3. $\bigwedge\limits_{i=1}^{k} \forall f \in \mathbf{F}_i^D \,\neg\Big(\exists f_1, \ldots, \exists f_{p^i} \in \mathbf{F}\,\Big(\Big(\bigwedge\limits_{q \neq s} f_q \neq f_s\Big) \wedge f_1 g = f_1 \wedge \ldots \wedge f_{p^i} g = f_{p^i} \wedge fhf_1 = fh = hf_1 \neq 0 \wedge \ldots \wedge fhf_{p^i} = fh = hf_{p^i} \neq 0\Big)\Big)$; this means that for every $i = 1, \ldots, k$ the inverse image of every $fA \subset D$, where $f \in \mathbf{F}_i^D$, contains at most $p^i - 1$ distinct elements $f_m A$ such that $f_m A \subset gA$ and $f_m \in \mathbf{F}$;

4. $\forall f \in \mathbf{F}^D\, \exists f' \in \mathbf{F}\,(f'g = g' \wedge hf' = fhf' \neq 0 \Rightarrow \exists \tilde{f} \in \mathbf{F}\,(\tilde{f}g = \tilde{f} \wedge h\tilde{f} = fh\tilde{f} \neq 0) \wedge \forall f' \in \mathbf{F}\,(f'g = f' \wedge hf' = fhf' \neq 0 \Rightarrow \exists \alpha\,(\alpha hf' = h\tilde{f})))$; this means that for every element $f \in \mathbf{F}$ either the inverse image fA is empty or it contains $\tilde{f} \in \mathbf{F}$ such that $\tilde{f}A \subset gA$ and the kernel of $\tilde{f}A$ under the mapping h has the maximal order of all kernels of $f'A$ for $f' \in \mathbf{F}$ such that $f'A \subset gA$;

5. $\neg\Big(\exists f_1, \ldots, f_p \in \mathbf{F}\,\Big(\Big(\bigwedge\limits_{q \neq s} f_q \neq f_s\Big) \wedge f_1 g = f_1 \wedge \ldots \wedge f_p = gf_p \wedge hf_1 = fhf_1 \neq 0 \wedge \ldots \wedge hf_p = fhf_p \neq 0 \wedge \Big(\bigwedge\limits_{q \neq s} f_q \sim_h f_s\Big)\Big)\Big)$, this means that there exist at most $p-1$ projections from $\mathbf{F}$ onto subgroups from gA such that their images are included in $\mathbf{F}^D$ and their kernels are isomorphic.

Two elements h_1 and h_2 from the set End_g are said to be equivalent ($h_1 \sim h_2$) if and only if the formula

$$\exists f_1\, \exists f_2\,((gf_1g) \cdot (gf_2g) = (gf_2g) \cdot (gf_1g) = g \wedge \forall f \in \mathbf{F}\,(fg = f \Rightarrow h_1 f = (gf_1g)h_2 f \wedge h_2 f = (gf_2g)h_1 f))$$

is true, i.e., there exist mutually inverse automorphisms gf_1g and gf_2g of gA which map h_2 and h_1 to automorphisms $(gf_1g)h_2$ and $(gf_2g)h_1$ such that we have $h_2 = (gf_2g)h_1$ and $h_1 = (gf_1g)h_2$ on gA for every projection from $\mathbf{F}$ that project onto a subgroup of gA.

Again the obtained set $\mathrm{End}_g /\!\sim$ will be denoted by $\widetilde{\mathrm{End}}_g$.

Every class $h \in \widetilde{\mathrm{End}}_g$ can be mapped to a set consisting of the following $k+1$ components: its ith component (where $i = 1, \ldots, k$) is a set of pairs

$$M_i = \{\langle f, m(f)\rangle \mid f \in \mathbf{F}_i^D,\ m = 1, \ldots, p^i - 1\},$$

where m is the dimension of the inverse image $f \in \mathbf{F}_i^D$, if it is not equal to zero; its $(k+1)$th component is a set of sequences

$$M = \{\langle f, m(f), l_0(f), \ldots, l_{m(f)}(f)\rangle \mid f \in \mathbf{F}^D,\ m \in \mathbb{N},\ l_1, \ldots, l_m = 0, \ldots, p-1\},$$

where p^m is the maximal order of the kernel of the inverse image of fA which is included in gA and has the form $f'A$ for $f' \in \mathbf{F}'$, l_i is the number of those inverse images of fA that belong to gA, have the form $f'A$ for $f' \in \mathbf{F}'$, and their kernels have the order p^i under h.

It is clear that endomorphisms from one equivalence class are mapped to the same sets $M_1, \ldots, M_k, M$, and endomorphisms from different classes are mapped to different sets. Further, every sequence $M_1, \ldots, M_k, M$ of finite sets of the described form are mapped to some class from $\widetilde{\mathrm{End}}_g$. Such a sequence of sets $M_1, \ldots, M_k, M$ is mapped to the element

$$\sum_{\langle f, m(f)\rangle \in M_1} m(f)a(f) + \cdots + \sum_{\langle f, m(f)\rangle \in M_k} m(f)a(f) + \sum_{\langle f, m(f), l_0, \ldots, l_{m(f)}\rangle \in M} l_1 c_1(f) + \cdots + l_{m(f)} c_{m(f)}(f),$$

where $a(f)$ is a fixed generator of the cyclic group fA for $f \in \mathbf{F}_1 \cup \cdots \cup \mathbf{F}_k$, and $c_1(f), \ldots, c_n(f), \ldots$ are fixed generators of the quasicyclic group fA for $f \in \mathbf{F}$.

Therefore we obtain a bijection between the set $\widetilde{\mathrm{End}}_g$ and the group A.

The addition ($h_3 = h_1 \oplus h_2$) on the set $\widetilde{\mathrm{End}}_g$ is introduced by a formula which is similar to the union of the formulas from Secs. 5.3 and 6.2, so we shall not write it here.

Proposition 3. *Let $A_1 = D_1 \oplus G_1$, $A_2 = D_2 \oplus G_2$, the group D_1 and D_2 be divisible and infinitely generated, the groups G_1 and G_2 be bounded by the number p^k, $|D_1| \geq |G_1|$, and $|D_2| \geq |G_2$. Then* $\mathrm{End}(A_1) \equiv \mathrm{End}(A_2) \Rightarrow A_1 \equiv_{\mathcal{L}_2} A_2$.

Proof. The proof is completely similar to the proof of Proposition 2, therefore we shall not write it here. □

6.4 Direct Sums of Divisible p-Groups and Bounded p-Groups of Greater Power

This case differs from the two previous cases, it is closer to the case of bounded p-groups, but it is more complicated.

We shall consider groups of the form $D \oplus G$, where D is an infinitely generated divisible group, G is a group bounded by the number p^k, $|G| > |D|$, $G = \sum_{i=1}^{k} G_i$, $G_i \cong \bigoplus_{\mu_i} \mathbb{Z}(p^i)$, $\mu_l = \max_{i=1,\dots,k} \mu_i$, and $D \cong \bigoplus_{\mu} \mathbb{Z}(p^\infty)$, $\mu < \mu_l$.

Assume that (as in the previous section) we have idempotents ρ_D and ρ_G which are projections on the divisible and bounded parts of the group A, respectively; and also projections $\rho_1, \dots, \rho_k$, where $\rho_1 + \dots + \rho_k = \rho_G$, on the direct summands $G_1, \dots, G_k$. Further, assume that the summand G_l with the maximal power μ_l, equal to the power of the whole group A, is known.

As above, we introduce various definable sets:

1. the set $\mathbf{F} = \mathbf{F}(\bar{g})$ consists of μ indecomposable projections on linearly independent direct summands of D;

2. for every $i = 1, \dots, k$ the set $\mathbf{F}_i = \mathbf{F}_i(\bar{g}_i)$ consists of μ_i projections on independent indecomposable direct summands of the group $G_i = \rho_i A$;

3. the set $\mathbf{F}' = \mathbf{F}'(\bar{g}')$ consists of μ_l projections on linearly independent countably generated direct summands of G_l;

4. an idempotent γ satisfies the following condition:

$$\Gamma(\gamma) := (\gamma\rho_l = \gamma \wedge \gamma^2 = \gamma \wedge \forall f \in \mathbf{F}' \, \exists \beta \, (f\gamma = \gamma f = \beta \wedge \mathrm{Idem}^\omega(\beta))).$$

   This condition means that γ is a projection on such a direct summand in G_l that its intersection with every subgroup fA, where $f \in \mathbf{F}'$, is a countably generated summand of G_l;

5. for every idempotent γ satisfying the formula $\Gamma(\gamma)$, by Γ_γ we shall denote the set $\{f \in \mathbf{F}_l \mid f\gamma = f\}$, and by $\Gamma_\gamma(g)$ for $g \in \mathbf{F}'$ we shall denote the set $\{f \in \mathbf{F}_l \mid f\gamma = f \wedge fg = f\}$. Let us fix two of these idempotents γ_0 and γ_1 with the conditions: (1) $\Gamma_{\gamma_0} \cap \Gamma_{\gamma_1} = \varnothing$; (2) for every $g \in \mathbf{F}'$ the set $\mathbf{F}_l \setminus (G_{\gamma_0} \cup G_{\gamma_1}) \cap \{f \in \mathbf{F}_l \mid fg = f\}$ is countable.

Denote Γ_{γ_0} by Γ_0 and Γ_{γ_1} by Γ_1.

6. Fix an endomorphism $\varphi \in \mathrm{End}(A)$ satisfying the following formula:

$$\begin{aligned}
\Phi(\varphi) := {} & \forall h \, (\mathrm{Idem}(h) \wedge \forall g \in \mathbf{F}' \, (hg = gh = 0) \Rightarrow \varphi h = h\varphi = 0) \\
& \wedge \forall g \in \mathbf{F}' \, (\forall f \in \Gamma_0(g) \, (\varphi f = f \wedge \forall f \in \Gamma_1(g) \, \exists f' \in \mathbf{F}_l \, (f' \notin \Gamma_1(g) \wedge f' \notin \Gamma_0(g) \\
& \wedge f'g = f' \wedge f'\varphi f = \varphi f \neq 0) \wedge \forall f \in \mathbf{F}_l \, (f \notin \Gamma_0(g) \wedge fg = f \\
& \Rightarrow \exists f' \in \mathbf{F}_l \, (f' \neq f \wedge f' \notin \Gamma_0(g) \wedge f' \notin \Gamma_1(g) \wedge f'g = f' \wedge f'\varphi f = \varphi f \neq 0) \\
& \wedge \forall f_1, f_2 \in \mathbf{F}_l \, (f_1 \neq f_2 \wedge f_1 g = f_1 \wedge f_2 g = f_2 \\
& \Rightarrow \neg(\exists f' \in \mathbf{F}_l \, (f'g = f' \wedge f'\varphi f_1 = \varphi f_1 \wedge f'\varphi f_2 = \varphi f_2)) \\
& \wedge \forall f' \in \mathbf{F}_l \, (f'g = f' \wedge f' \in \Gamma_1(g) \Rightarrow \exists f \in \mathbf{F}_l \, (fg = f \wedge f'\varphi = f'\varphi f = \varphi f)) \\
& \wedge \forall h \, (\mathrm{Idem}(h) \wedge hg = h \wedge h\gamma_0 = \gamma_0 h = 0 \wedge \forall f \, (\mathrm{Idem}^*(f) \wedge fg = f \wedge fh = f \\
& \Rightarrow \exists f' \, (\mathrm{Idem}^*(f') \wedge f'g = f' \wedge f'h = f' \wedge f'\varphi f = \varphi f) \Rightarrow \mathrm{Idem}^\omega(f)))).
\end{aligned}$$

This condition introduces an endomorphism φ that maps the complementary direct summand for $\sum_{g\in\mathbf{F}'} gA$ to 0, i.e., acts only on $\sum_{g\in\mathbf{F}'} gA$ in the following way: for every $g\in\mathbf{F}'$ the elements αA, where $\alpha\in\Gamma_0(g)$, are mapped into themselves, and the elements αA, where $\alpha\in\Gamma_1(g)$, are mapped somewhere to the elements of $\mathbf{F}_l A\setminus(\Gamma_0(g)A\cup\Gamma_1(g)A)$ which are included in gA. Further, φ is a monomorphism on gA, and its image is

$$\langle\Gamma_0(g)A\rangle\oplus\langle\{fA\mid f\in\mathbf{F}_l\wedge fg=f\wedge f\notin\Gamma_1(g)A\}\rangle.$$

Outside $\Gamma_0(A)$ there are no finite-dimensional proper subspaces of this endomorphism. Therefore, we can numerate all elements from $\mathbf{F}_l$ that project on subgroups from gA (we shall denote this set by $F_l(g)$) by the following: $f_i^j=f_i^j(g)$, $i=0,1,\dots$, $j=1,\dots$, and

a. $f_0^j\in\Gamma_0(g)$;

b. $f_1^j\in\Gamma_1(g)$;

c. $\varphi(f_i^jA)=f_{i+1}^jA$ if $i>0$;

d. $\varphi(f_0^jA)=f_0^jA$.

We shall denote the set $\{f_i^j\}_{j=1,\dots}$ by $\Gamma_i(g)$ (note that for an arbitrary i this set is not definable).

7. The union $\bigcup_{g\in\mathbf{F}'}\mathbf{F}_l(g)$ will be denoted by $\mathbf{F}_l'$. This set is definable;

8. note that on the group $B=\langle\mathbf{F}_l'A\rangle$ the endomorphism φ has a left inverse endomorphism ψ such that $\psi\circ\varphi=1_B$. For every g we shall introduce gA as follows:

$$\begin{cases}\psi(f_0^jA)=f_0^jA,\\ \psi(f_i^jA)=f_{i-1}^jA \text{ if } i>1,\\ \psi(f_1^jA) \text{ can be arbitrary.}\end{cases}$$

We shall consider ψ with the condition $\psi(f_i^jA)=0$. Then two elements $f_1,f_2\in\mathbf{F}_l(g)\setminus\Gamma_0(g)$ (or, more generally, $\mathbf{F}_l'\setminus\Gamma_0$) will be called *$\varphi$-equivalent* ($f_1\sim_\varphi f_2$) if

$$\begin{aligned}\exists h_1\,\exists h_2\,\exists\alpha\Big(&h_1g=h_1\wedge h_2g=h_2\wedge\mathrm{Idem}(h_1)\wedge\mathrm{Idem}(h_2)\wedge\alpha^2=1\\ &\wedge\bigwedge_{i=1}^{2}\forall f\,(\mathrm{Idem}^*(f)\wedge fg=f\wedge fh_i=f\Rightarrow\exists f'\,(\mathrm{Idem}^*(f')\wedge f'g=f'\wedge f'h_i=f'\wedge f'\psi f=\psi f))\\ &\wedge f_1h_1=f_1\wedge f_2h_2=f_2\wedge\bigwedge_{i=1}^{2}\forall h\,(\mathrm{Idem}(h)\wedge hg=h\wedge\forall f\,(\mathrm{Idem}^*(f)\wedge fg=f\wedge fh=f\\ &\Rightarrow\exists f'\,(\mathrm{Idem}^*(f')\wedge f'g=f'\wedge f'h=f'\wedge f'\psi f=\psi f)\wedge f_ihf_i\Rightarrow h_ih=h_i))\\ &\wedge h_1\alpha h_2=\alpha h_2=h_1\alpha\wedge h_2\alpha h_1=h_2\alpha\Big).\end{aligned}$$

This means that minimal proper subspaces of the endomorphism ψ (h_1A and h_2A), containing the groups f_1A and f_2A, respectively, have the same power, i.e., $f_1A,f_2A\in\operatorname{Ker}\psi^m\setminus\operatorname{Ker}\psi^{m+1}$ for some natural m, or $f_1A,f_2A\in\varphi^m(\Gamma_1)$. It is clear that in this case $f_1=f_m^j(g)$ and $f_2=f_m^i(g)$ (or, in more general case, $f_1=f_m^j(g_1)$ and $f_2=f_m^j(g_2)$).

We shall call an element $f_1\in\mathbf{F}_l(g)$ a *φ-successor* of the element $f_2\in\mathbf{F}_l(g)$ ($f_1\sim_\varphi f_2+1$) if

$$\exists f\,(f\sim_\varphi f_1\wedge f\varphi f_2=\varphi f_2=f\varphi).$$

A similar formula will introduce a notion of a *φ-greater element* ($f_1 >_\varphi f_2$) as an element for which a minimal proper subspace of an endomorphism ψ containing $f_1 A$ has greater power than the corresponding subspace for $f_2 A$.

Let us fix $g \in \mathbf{F}'_l$ and construct an interpretation of the group $A = D \oplus G$ for this g.

Consider the set End_g of all homomorphisms $h\colon gA \to A$ satisfying the following conditions:

1. $\forall f \in \mathbf{F}_l \Big(fg = f \Rightarrow \Big(hf = 0 \vee \exists \tilde{f} \in \mathbf{F}^D (hf = \tilde{f}hf \neq 0) \vee \Big(\bigvee_{i=1}^{k} \exists \tilde{f} \in \mathbf{F}_i (hf = \tilde{f}hf \neq 0)\Big)\Big)\Big)$;

2. $\exists f\, (\mathrm{Fin}(f) \wedge \mathrm{Idem}(f) \wedge fh = h)$;

3. $\bigwedge_{i=1}^{k} \forall f \in \mathbf{F}_i \neg\Big(\exists f_1, \ldots, \exists f_{p^i} \in \mathbf{F} \Big(\bigwedge_{q \neq s} f_q \neq f_s \wedge f_1, \ldots, f_{p^i} \in \Gamma_0(g) \wedge hf_1 = fhf_1 \neq 0 \wedge \ldots \wedge hf_{p^i} = fhf_{p^i} \neq 0\Big) \wedge \forall f' \in \mathbf{F}_l (hf' = fhf' \neq 0 \Rightarrow f' \in \Gamma_0(g))\Big)$; this means that the inverse images of the elements of the bounded summand can be contained only in the set $\Gamma_0(g)$;

4. $\forall f \in \mathbf{F}^D \exists f' \in \mathbf{F}_l (hf' = fhf' \neq 0 \Rightarrow f' \notin \Gamma_0)$; contrary to the previous assertion, this means that the inverse images of the elements of the divisible summand can not be contained in $\Gamma_0(g)$;

5. $\forall f \in \mathbf{F}^D \neg\Big(\exists f_1, \ldots, f_p \in \mathbf{F}_l \Big(\Big(\bigwedge_{q \neq s} f_q \neq f_s \wedge f_q \sim_\varphi f_s\Big) \wedge f_1 g = f_1 \wedge \ldots \wedge f_p g = f_p \wedge hf_1 = fhf_1 \neq 0 \wedge \ldots \wedge hf_p = fhf_p \neq 0\Big)\Big)$; this means that no element from $\mathbf{F}_D$ can have more than $p-1$ φ-equivalent inverse images;

6. $\forall f \in \mathbf{F}^D \exists f' \in \mathbf{F}_l \neg(\exists f'' \in \mathbf{F}_l (f'' >_\varphi f' \wedge hf'' = fhf'' \neq 0))$, i.e., every element from $\mathbf{F}_D$ contains only a finite number of inverse images in $\mathbf{F}_l$.

Two elements h_1 and h_2 of the set End_g are said to be equivalent (notation: $h_1 \sim h_2$) if we have the following formula:

$$\begin{aligned}
&\exists f_1 \exists f_2 ((gf_1g) \cdot (gf_2g) = (gf_2g) \cdot (gf_1g) = g \\
&\quad \wedge \forall f \in \mathbf{F}_l (fg = f \Rightarrow \forall f' \in \mathbf{F}^D \cup \mathbf{F}_1 \cup \cdots \cup \mathbf{F}_k (h_1 f = f' h_1 f \neq 0 \Leftrightarrow (gf_1gh_2)f = f'(gf_1gh_2)f \neq 0)) \\
&\quad \wedge \forall f \in \mathbf{F}_l (gf_1g \cdot f \sim_\varphi f)).
\end{aligned}$$

The obtained set $\mathrm{End}_g/\!\sim$ will be denoted by $\widetilde{\mathrm{End}}_g$.

Now we shall show how to find a bijection between the set $\widetilde{\mathrm{End}}_g$ and the group A. Let us consider some $h \in \widetilde{\mathrm{End}}_g$. For every $f \in \mathbf{F}_i$ let us consider the intersection of the inverse image of fA with the set $\Gamma_0 A$. Suppose that this inverse image contains m_f elements. Thus, we get the set

$$M_G = \bigcup_{i=1}^{k} \{\langle f, m_f \rangle \mid f \in \mathbf{F}_i,\ m_f = 1, \ldots, p^i - 1\}.$$

For every $f \in \mathbf{F}^D$ and every natural j let us consider the intersection of the inverse image of fA with the set $\Gamma_j(g)$. Suppose that this inverse image contains l_f^j elements, and the maximal nonzero j is equal to γ_f. Then we get the set

$$M_D = \{\langle f, \gamma_f, l_f^1, \ldots, l_f^{\gamma_f} \rangle \mid f \in \mathbf{F}^D,\ \gamma_f \in \mathbb{N},\ l_f^1, \ldots, l_f^{\gamma_f} \in \{0, \ldots, p-1\}\}.$$

Now an element h will be mapped to the following sum:

$$\sum_{f \in M_G} m_f a(f) + \sum_{f \in M_D} l_f^1 c_1(f) + \cdots + l_f^{\gamma_f} c_{\gamma_f}(f) \in A.$$

It is clear that such a mapping is a bijection between the sets $\widetilde{\mathrm{End}}_g$ and A, and this bijection becomes an isomorphism if and only if we introduce addition on the set $\widetilde{\mathrm{End}}_g$ with the help of a formula similar to the formulas from Secs. 5.3 and 6.2. Therefore we have the following proposition.

Proposition 4. *Let $A_1 = D_1 \oplus G_1$, $A_2 = D_2 \oplus G_2$, the groups D_1 and D_2 be divisible, the groups G_1 and G_2 be infinite and bounded by the number p^k, $|D_1| < |G_1|$, and $|D_2| < |G_2|$. Then* $\mathrm{End}(A_1) \equiv \mathrm{End}(A_2) \Rightarrow A_1 \equiv_{\mathcal{L}_2} A_2$.

7 Groups with Unbounded Basic Subgroups

7.1 The Case $A = D \oplus G$, Where $|D| \geq |G|$, and Other Cases

Let us separate our problem into three cases.

1. $A = D \oplus G$, where $|D| \geq |G|$ and G is any unbounded group. We shall not consider this case in details, because its proof is similar to the proof from Sec. 6.2.

This case resembles the case $A = D \oplus G$, where $|D| \geq |G|$, D is a divisible group, and G is a bounded group (see Sec. 6.3). Here we give only a sketch of the proof.

Since $|G| \leq |D|$, we have that a basic subgroup of the group G (or the group A) is of power not greater than the power of D. Hence there exists an embedding $\varphi_1 \colon B \to D_1$, where $D = D_1 \oplus D_2 \oplus D_3$ and $|D| = |D_1| = |D_2| = |D_3|$. This embedding will be fixed, and after that we can assume that the group B is a subgroup of D_1. Further, $|G| \leq |D|$ implies $|G/B| \leq |D|$, whence there exists an embedding $\varphi_2 \colon G/B \to D_2$ (i.e., a mapping from G into D_2 which is equal to zero on B). Thus the group G/B can also be considered as a subgroup of D_2.

Now we shall find some definable sets:

1. the set $\mathbf{F}_1$ consists of $|B|$ independent indecomposable projections on quasicyclic direct summands in a minimal direct summand of D_1, containing $\varphi_1(B)$ as a subgroup;

2. the set $\mathbf{F}_2$ consists of $|G/B|$ independent indecomposable projections on quasicyclic direct summands of $\varphi_2(G/B)$;

3. the sets $\mathbf{F}$ and $\mathbf{F}_3$ consists of $\mu = |D|$ independent projections on quasicyclic direct summands of the groups D and D_3, respectively;

4. the set $\mathbf{F}'$ consists of μ independent projections on countably generated direct summands of the group D.

For every $g \in \mathbf{F}'$ an interpretation of the group A will be constructed in the following way: we shall consider homomorphisms $h \colon gA \to A$ such that the images of the subgroups fA ($f \in \mathbf{F}$, $fA \subset gA$) are either zero or are contained in $f'A$ ($f' \in \mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3$), and $h(gA)$ is finite-dimensional.

The inverse images of fA, where $f \in \mathbf{F}_1$, will interpret the summands from B in the decomposition of the element $a \in A$ in a quasibasis; the inverse images of fA, where $f \in \mathbf{F}_2$, are the summands from G/B, i.e., $c_{i,j}$ for $i \in \omega$, $j \in |G/B|$; the inverse images of fA, where $f \in \mathbf{F}_3$, are the summands from D. The rest of the procedure is similar to that from the previous sections.

So we have given a sketch of the proof of the following proposition.

Proposition 1. *Let $A_1 = D_1 \oplus G_1$, $A_2 = D_2 \oplus G_2$, where the groups D_1, D_2 are divisible, the groups G_1, G_2 are reduced, $|D_1| \geq |G_1|$, and $|D_2| \geq |G_2|$. Then* $\mathrm{End}(A_1) \equiv \mathrm{End}(A_2) \Rightarrow A_1 \equiv_{\mathcal{L}_2} A_2$.

In this section, we shall assume that $A = D \oplus G$, where $|D| < |G|$.

2. $A = D \oplus G$, where $|D| < |G|$, B is a basic subgroup in G, and $r(B) = r_{\mathrm{fin}}(B)$.

The case $r_{\mathrm{fin}}(B) > \omega$ will be considered in Sec. 7.4, and the case $r_{\mathrm{fin}} = \omega$ will be considered in Sec. 7.5.

If $r(B) > \omega$, then $|A| = r(B)$ and $|D| < r(B)$. If $r(B) = \omega$, then $|A| \leq |\mathcal{P}(\omega)|$, therefore if we do not assume the continuum-hypothesis, then we can meet the situation, where $\omega < |D| < |A| \leq 2^\omega$, which is not good for us. Thus for the simplicity of arguments, we shall assume the continuum-hypothesis.

Hence, if $A = D \oplus G$, where $|D| < |G|$, $r(B) = r_{\mathrm{fin}}(B)$, then we shall interpret the theory $\mathrm{Th}_2^{r(B)}(A)$ in the ring $\mathrm{End}(A)$.

3. $A = D \oplus G$, where $|D| < |G|$ and $r(B) \neq r_{\mathrm{fin}}(B)$. If in this case $r_{\mathrm{fin}}(B) > \omega$, then we can obtain the full second order theory of the group A. This case is considered in Sec. 7.4.

If $r_{\mathrm{fin}}(B) = \omega$, then, assuming the continuum-hypothesis, we can find in the group A a bounded direct summand of power $|A|$, and in this case we can define the complete second order theory of the group A. This case is considered in Sec. 7.6.

In Secs. 7.2 and 7.3, we shall find some definable objects, which are important for all cases.

7.2 Definable Objects

In this section, we assume that $A = D \oplus G$, the group D is divisible (it can be zero), the group G is reduced and has an unbounded basic subgroup B,

$$B = B_1 \oplus \dots \oplus B_n \oplus \dots,$$

where

$$B_n \cong \bigoplus_{\mu_n} \mathbb{Z}(p^n),$$

$r(D) = \mu_D$, $|B| = \bigcup_{n \in \mathbb{N}} \mu_n = \mu_B$, $|G| = \mu_G$ (if $\mu_B > \omega$, then $\mu_G = \mu_B$), and $\mu = |A| = \max(\mu_D, \mu_G)$.

We suppose that projections ρ_D and ρ_G on the summands D and G of the group A, respectively, are fixed.

By Z we shall denote the center of the ring $\mathrm{End}(A)$. As we remember (see Theorem 12), each of its elements multiplies all elements of A by some fixed p-adic number.

For any indecomposable projections ρ_1 and ρ_2 from $\mathrm{End}(G)$ we shall write $o(\rho_1) \le o(\rho_2)$ if

$$\forall c \in Z\, (c\rho_2 = 0 \Rightarrow c\rho_1 = 0).$$

It is clear that this formula holds if and only if the order of the finite cyclic direct summand $\rho_1 A$ is not greater than the order of the summand $\rho_2 A$.

Similarly,

$$(o(\rho_1) < o(\rho_2)) := (o(\rho_1) \le o(\rho_2)) \wedge \neg(o(\rho_2) \le o(\rho_1)),$$
$$(o(\rho_1) = o(\rho_2)) := (o(\rho_1) \le o(\rho_2)) \wedge (o(\rho_2) \le o(\rho_1)).$$

For every indecomposable projection ρ we shall consider the following formula sets.

1. The formula

$$\mathrm{Ord}_\rho(f) := \mathrm{Idem}(f) \wedge \forall f'\, (\mathrm{Idem}^*(f') \wedge f'f = f' \Rightarrow o(f') = o(\rho))$$

defines the projections f on direct summands fA in A which are direct sums of cyclic groups of order $o(\rho A)$.

2. The formula

$$\mathrm{MaxOrd}_\rho(f) := \mathrm{Idem}(f) \wedge \mathrm{Ord}_\rho(f) \wedge \forall f'\, (\mathrm{Ord}_\rho(f') \Rightarrow \neg(ff' = f))$$

defines the projections f on maximal direct summands fA in A which are direct sums of cyclic groups of order $o(\rho A)$.

3. The formula

$$\mathrm{Rest}_\rho(f) := \mathrm{Idem}(f) \wedge \forall f'\, (\mathrm{Idem}^*(f') \wedge f'f = f' \Rightarrow o(f') \le o(\rho))$$

defines the projections f on direct summands fA in A which are direct sums of cyclic groups of order at most $o(\rho A)$.

4. The formula

$$\mathrm{MaxRest}_\rho(f) := \mathrm{Idem}(f) \wedge \mathrm{Ord}_\rho(f) \wedge \forall f' \,(\mathrm{Ord}_\rho(f') \Rightarrow \neg(ff' = f))$$

defines the projections f on maximal direct summands fA in A which are direct sums of cyclic groups of order at most $o(\rho A)$.

5. The formula

$$\overline{\mathrm{Base}}(\varphi) := \forall \rho \, \exists f \,(\mathrm{MaxRest}_\rho(f) \wedge \forall f' \,(\mathrm{Idem}^*(f') \Rightarrow (f'f = f' \Leftrightarrow \forall c \in Z \,(cf' \neq 0 \Rightarrow c(f'\varphi) \neq 0)))$$

postulates that for every natural n there exists a maximal p^n-bounded direct summand of the group A which is included in φA. Therefore, the group φA necessarily contains some basic subgroup of the group A.

6. The formula

$$\begin{aligned}\mathrm{Base}(\varphi) := \overline{\mathrm{Base}}(\varphi) \wedge \forall f^* \,(\mathrm{Idem}^*(f^*) \wedge f^*\varphi \neq 0 \Rightarrow \exists \rho \, \exists f \,(\mathrm{MaxRest}_\rho(f) \\ \wedge \forall f' \,(\mathrm{Idem}^*(f') \Rightarrow (f'f = f' \Leftrightarrow \forall c \in Z \,(cf' \neq 0 \Rightarrow c(f\varphi) \neq 0))) \wedge f^*f = f^*))\end{aligned}$$

is true for every endomorphism $\varphi \in \mathrm{End}(A)$ whose image is a basic subgroup in A.

Let us suppose that we have a fixed endomorphism φ_B such that $\mathrm{Base}(\varphi_B)$.

7.3 Definable Special Sets

We shall consider three different cases:

1. $\mu_B = \omega$;
2. $\mu_B > \omega$ and $\forall k \in \mathbb{N} \, \exists n \in \mathbb{N} \,(n > k \wedge \mu_n = \mu_B)$. This is always true if $\mathrm{cf}\mu_B > \omega$;
3. $\mu_B > \omega$, $\mathrm{cf}\mu_B = \omega$, $\forall n \in \mathbb{N} \,(\mu_n < \mu_B)$.

Case 1. $\mu_B = \omega$.

Let us consider the formula

$$\begin{aligned}\mathrm{Intr}(f) := [\forall f' \,(\mathrm{Idem}(f') \wedge f'\varphi_B \neq 0 \Rightarrow f'f \neq 0)] \wedge [\forall f_1 \, \forall f_2 \,(\mathrm{Idem}^*(f_1) \wedge \mathrm{Idem}^*(f_2) \wedge o(f_1) = o(f_2) \\ \wedge \forall c \in Z \,(cf_1 \neq 0 \Rightarrow cf_1 f \neq 0) \wedge \forall c \in Z \,(cf_2 \neq 0 \Rightarrow cf_2 f \neq 0) \Rightarrow f_1 f_2 \neq 0 \wedge f_2 f_1 \neq 0)] \\ \wedge [\forall \rho' \,(\mathrm{Idem}^*(\rho') \Rightarrow \exists f' \,(\mathrm{Idem}^*(f') \wedge o(f') > o(\rho') \wedge \forall c \in Z \,(cf' \neq 0 \Rightarrow cf'f \neq 0)))].\end{aligned}$$

The first part of this formula, enclosed in square brackets, postulates $fA \subset \varphi_B A = B$. The second part states that there is at most one cyclic direct summand of the same order in the image of fA. The third part states that the orders of direct summands in fA are unbounded.

Therefore this formula gives us an endomorphism f with image B' being a cyclic direct summand in B,

$$B' \cong \bigoplus_{i \in \mathbb{N}} \mathbb{Z}(p^{n_i}),$$

where (n_i) is an increasing sequence.

This endomorphism is supposed to be fixed and is denoted by f_B.

Now we shall consider endomorphisms from B' into A. Namely we shall consider only functions f with the condition

$$\forall \rho \,(\mathrm{Idem}(\rho) \wedge \rho f_B = 0 \Rightarrow \rho f = 0).$$

Two functions f_1 and f_2 satisfying this condition are said to be equivalent if

$$\forall \rho \,(\mathrm{Idem}^*(\rho) \wedge \forall c \in Z \,(c\rho \neq 0 \Rightarrow c\rho f \neq 0) \Rightarrow f_1\rho = f_2\rho),$$

i.e., if they coincide on the group B'.

Consequently, if we factorize the set of all described functions by this equivalence, then we obtain the group $\mathrm{Hom}(B', A)$.

Introduce now the formula

$$o(\rho_1) \geq o(\rho_2)^2$$

for two indecomposable idempotents ρ_1 and ρ_2 as follows:

$$\forall c \in Z\,(c\rho_2 \neq 0 \Rightarrow c^2\rho_1 \neq 0).$$

This formula means that $|\rho_1 A| \geq |\rho_2 A|^2$.

Similarly we can introduce the formulas

$$o(\rho_1) > o(\rho_2)^2 \ \text{ and } \ o(\rho_1) = o(\rho_2)^2.$$

Suppose now that our function f_B satisfies the additional condition

$$\begin{aligned}&\forall f'\,(\mathrm{Idem}^*(f') \wedge \forall c \in Z\,(cf' \neq 0 \Rightarrow cf'f_B \neq 0) \Rightarrow pf' \neq 0)\\ &\quad \wedge \forall \rho'\,\forall f'\,(\mathrm{Idem}^*(f') \wedge \forall c \in Z\,(cf' \neq 0 \Rightarrow cf'f_B \neq 0) \wedge o(f') = o(\rho')\\ &\quad \Rightarrow \forall f\,(\mathrm{Idem}^*(f) \wedge \forall c \in Z\,(cf \neq 0 \Rightarrow cff_B \neq 0) \wedge o(f) > o(\rho') \Rightarrow o(f) > o(\rho')^2)).\end{aligned}$$

This condition means that

1. every cyclic direct summand in B' of the smallest order has order greater than p (i.e., at least not smaller than p^2);
2. for every direct cyclic summand in B' of order p^k the next cyclic summand of greater order has order greater than p^{2k}.

Therefore,

$$B' \cong \bigoplus_{i\in\omega} \mathbf{Z}(p^{n_i}),$$

where $n_1 \geq 2$, $n_{i+1} > 2n_i$.

Now consider the formula

$$\begin{aligned}&\mathrm{Ins}(\psi) := [\exists f\,(\mathrm{Idem}^*(f) \wedge \forall c \in Z\,(cf \neq 0 \Rightarrow cff_B \neq 0)\\ &\quad \wedge \forall f'\,(\mathrm{Idem}^*(f') \wedge \forall c \in Z\,(cf' \neq 0 \Rightarrow cf'f_B \neq 0) \Rightarrow o(f) \leq o(f')) \wedge \psi f = pf)]\\ &\quad \wedge [\forall f_1\,\forall c_1 \in Z\,(\mathrm{Idem}^*(f_1) \wedge \forall c \in Z\,(cf_1 \neq 0 \Rightarrow cf_1 f_B \neq 0)\\ &\quad \wedge (\psi f_1 = c_1 f_1 \Rightarrow \exists f_2\,(\mathrm{Idem}^*(f_2) \wedge \forall c \in Z\,(cf_2 \neq 0 \Rightarrow cf_2 f_B \neq 0) \wedge o(f_2) > o(f_1)\\ &\quad \wedge \forall f'\,(\mathrm{Idem}^*(f') \wedge \forall c \in Z\,(cf' \neq 0 \Rightarrow cf'f_B \neq 0) \Rightarrow o(f') \leq o(f_1) \vee o(f') \geq o(f_2)) \wedge \psi f_2 = pc_1 f_2)))].\end{aligned}$$

The condition from the first square brackets states that there exists a cyclic summand of the smallest order such that the action of the endomorphism ψ on it is multiplication by p. The second condition states that for every natural i there exists a direct cyclic summand $\langle a_i \rangle$ of order p^{n_i} such that the action of ψ on it is multiplication by p^i.

Suppose that for some different cyclic direct summands $\langle a_i \rangle$ and $\langle b_i \rangle$ from B' the action of ψ on them is multiplication by p^i. Let $b_i = \sum \alpha_k a_k + \sum \beta_l a_l + a_i$, where $o(a_k) < o(b_i)$, $o(a_l) > o(b_i)$, $k < i$, $l > i$,

$$\psi(b_i) = p^i b_i = \sum p^k \alpha_k a_k + \sum p^l \beta_l a_l + p^i a_i = \sum p^i \alpha_k a_k + \sum p^i \beta_l a_l + p^i a_i.$$

Let $k < i$. Then we have $p^i \alpha_k a_k = p_k \alpha_k a_k = 0$, i.e., α_k is divisible by $p^{n_k - k}$ and $p^{n_k - k}$ is divisible by p^k. Therefore, we can write

$$b_i = \sum \alpha_k p^k a_k + \sum \beta_k p^{n_l - n_i} a_l + a_i.$$

We have $p^i\beta_l p^{n_l-n_i}a_l = p^l\beta_l p^{n_l-n_i}a_l = 0$. Note that every cyclic direct summand $\langle a\rangle$ either uniquely corresponds to an element of the center $c \in Z$ such that $\psi a = ca$, or there is no such element. We can consider only those summands that correspond to elements of the center.

Let us suppose that we have some homomorphism $f\colon B' \to A$ such that $o(f(a_i)) \leq p^i$. Let $\psi(a_i) = p^i a_i$ and $\psi(b_i) = p^i b_i$. Let us find $f(b_i)$:

$$f(b_i) = \sum \alpha_k p^k f(a_k) + \sum \beta_l p^{n_l-n_i} f(a_l) + f(a_i).$$

Since $o(f(a_k)) \leq p^k$, we have $\sum \alpha_k p^k f(a_k) = 0$. Since $o(f(a_l)) \leq p^l$, we have $\sum \beta_l p^{n_l-n_i} f(a_l) = 0$. Therefore $f(b_i) = f(a_i)$. Thus every element of the center Z of the form $p^n \cdot E$ is mapped (under this homomorphism $f\colon B' \to A$) to some uniquely defined element $a \in A$ with the condition $o(a) \leq p^n$.

Case 2. $\forall k \in \omega\, \exists n \in \omega\, (n > k \wedge \mu_n = \mu_B)$.

Consider the formula

$$\begin{aligned}\mathrm{ECard}(\rho) :=\ & \mathrm{Idem}^*(\rho) \wedge \exists \psi\, \forall f\, (\mathrm{Idem}^*(f) \wedge \forall c \in Z\, (cf \neq 0 \Rightarrow cf\varphi_B \neq 0)\\ & \Rightarrow \exists f'\, (\mathrm{Idem}^*(f') \wedge \forall c \in Z\, (cf' \neq 0 \Rightarrow cf'\varphi_B \neq 0) \wedge o(f') = o(\rho) \wedge f\psi f' \neq 0)).\end{aligned}$$

This formula states that the set of independent cyclic summands of order $o(\rho)$ has the same power as the whole group B, because there exists a homomorphism ψ from a direct summand of the group B (which is isomorphic to the sum of cyclic groups of order $o(\rho)$) such that its image intersects with every cyclic summand of B. Therefore, $\mu_{o(\rho)} = \mu_B$.

Now let us consider the formula

$$\begin{aligned}\mathrm{Fine}(f) :=\ & [\forall f'\, (\mathrm{Idem}(f') \wedge f'\varphi_B \neq 0 \Rightarrow f'f \neq 0)]\\ & \wedge [\forall f_1\, \forall f_2\, (\mathrm{Idem}^*(f_1) \wedge \mathrm{Idem}^*(f_2) \wedge o(f_1) = o(f_2)\\ & \wedge \forall c \in Z\, (cf_1 \neq 0 \Rightarrow cf_1 f \neq 0) \wedge \forall c \in Z\, (cf_2 \neq 0 \Rightarrow cf_2 f \neq 0) \Rightarrow f_1 f_2 \neq 0 \wedge f_2 f_1 \neq 0)]\\ & \wedge [\forall \rho'\, (\mathrm{ECard}(\rho') \Leftrightarrow \exists f'\, (\mathrm{Idem}^*(f') \wedge o(f') = o(\rho') \wedge \forall c \in Z\, (cf' \neq 0 \Rightarrow cf'f \neq 0)))].\end{aligned}$$

The first part of the formula, enclosed in square brackets, postulates $fA \subset \varphi_B A = B$. The second part, enclosed in the second square brackets, states that the image of fA does not contain more than one cyclic direct summand of any order. The third part states that all direct cyclic summands have order p^n, where $\mu_n = \mu_B$.

To make the further constructions, we need to recall Sec. 3.

Formulation of Theorem 1 need not be changed, but Lemma 5 and the proof of the theorem with the help of this lemma must be changed a little for our case.

There is a new formulation of Lemma 5: *there exists a formula $\varphi(f)$ with one free variable f such that $\varphi(f)$ holds in $\mathrm{End}(B')$ if and only if there exists an ordinal number $\alpha \in \mu$ such that for all $\beta \in \mu$ and all $m \in \omega$*

$$f(a^m_{\langle 0,\beta\rangle}) = a^m_{\langle \alpha,\beta\rangle}.$$

Now we shall write the proof of the theorem with the help of the lemma.

Let a function f_0^* map (for every $m \in \omega$) the set $\{a^m_{\langle 0,\alpha\rangle} \mid \alpha \in \mu\}$ onto the set $\{a^m_t \mid t \in I^*\}$, and $f_0^*(a^m_{\langle \alpha,\beta\rangle}) = f_0^*(a^m_{\langle 0,\beta\rangle})$. Suppose that we have a set $\{f_i\}_{i\in\mu}$ and let the function f^* be such that

$$f^*(a^m_{\langle \alpha,\beta\rangle}) = f_\alpha \circ f_0^*(a^m_{\langle \alpha,\beta\rangle}).$$

Let f_1^* map (for every $m \in \omega$) the set $\{a^m_t \mid t \in I^*\}$ onto the set $\{a^m_{\langle 0,\beta\rangle} \mid \beta \in \mu\}$. Let the formula $\tilde\varphi(f', \dots)$ say that there exists f such that

1. $\varphi(f)$;
2. $f' \circ f_0^* \circ f_1^* = f^* \circ f \circ f_1^*$.

Then $\mathrm{End}(B') \vDash \varphi(f)$ if and only if there exists $\alpha \in \mu$ such that

$$\forall \beta \in \mu \, \forall m \in \omega \, f(a^m_{\langle 0,\beta\rangle}) = a^m_{\langle \alpha,\beta\rangle}.$$

Therefore

$$f' \circ f_0^* \circ f_1^*(a_t^m) = f^* \circ f \circ f_1^*(a_t^m) \Leftrightarrow f' \circ f_0^*(a^m_{\langle 0,\beta\rangle}) = f^* \circ f(a^m_{\langle 0,\beta\rangle})$$
$$\Leftrightarrow f' \circ f_0^*(a^m_{\langle 0,\beta\rangle}) = f^*(a^m_{\langle \alpha,\beta\rangle}) \Leftrightarrow f' \circ f_0^*(a^m_{\langle 0,\beta\rangle}) = f_\alpha \circ f_0^*(a^m_{\langle 0,\beta\rangle}).$$

Let $f_0^*(a^m_{\langle 0,\beta\rangle}) = a^m_{t_\beta}$. Then

$$f'(a^m_{t_\beta}) = f_\alpha(a^m_{t_\beta}),$$

what we needed.

Now we need to change the proof of the lemma. The case $\mu = \omega$ is not interesting for us, therefore we shall begin from the second case.

Suppose that the cardinal number μ_B is regular. Represent it in the form of the union of the sets I_0, I, and J, where $|I_0| = |J| = |I| = \mu_B$, $J = \{\langle \alpha,\beta\rangle \mid \alpha,\beta \in \mu\} \cup \{0\}$, $I = \{\langle \alpha,\delta,n\rangle \mid \alpha \in \mu,\ \delta \in \mu,\ \mathrm{cf}\delta = \omega,\ n \in \omega\}$, and $a_\alpha^{\beta,n} = a_{\langle \alpha,\beta,n\rangle}$. As in Sec. 3, for every limit ordinal $\delta \in \mu$ such that $\mathrm{cf}\delta = \omega$, we choose an increasing sequence $(\delta_n)_{n\in\omega}$ of ordinal numbers less than δ such that their limit is δ and for each $\beta \in \mu$ and $n \in \omega$ the set $\{\delta \in \mu \mid \beta = \delta_n\}$ is a stationary subset in μ.

Consider an independent set of generators of cyclic direct summands from B such that

1. the order of every generator from this set is equal to p^n, where $\mu_n = \mu_B$;
2. for every $n \in \omega$ such that $\mu_n = \mu_B$, the set of all elements of order p^n from this set has the power μ_B.

Let us denote this set by

$$\{a_t^n \mid t \in J \cup I_0 \cup I = I^*,\ o(a_t^n) = p^n,\ \mu_n = \mu_B\}.$$

Let us define the functions $f_0^*, \dots, f_{14}^*$, similar to the functions from the case II from Sec. 3, but with some addition: $f_0^*(a_t^m) = a_0^m$, $f_1^*(a_t^m) = a_t^{m-1}$ for $m > 0$, $f_1^*(a_t^0) = 0$, $f_2^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle 0,0\rangle}$, $f_3^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle \alpha,0\rangle}$, $f_4^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle 0,\beta\rangle}$, $f_5^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle \beta,\alpha\rangle}$, $f_6^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle \alpha,\alpha\rangle}$, for $\delta \in \mu_B$, $\mathrm{cf}\delta = \omega$, $f_7^*(a^m_{\langle \alpha,\delta\rangle}) = a^m_{\langle \alpha,\delta,0\rangle}$, for $\delta \in \mu_B$, $\mathrm{cf}\delta \neq \omega$, $f_7^*(a^m_{\langle \alpha,\delta\rangle}) = a^m_{\langle \alpha,\delta\rangle}$, $f_8^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle \alpha,\beta\rangle}$, $f_8^*(a^m_{\langle \alpha,\delta,n\rangle}) = a^m_{\langle \alpha,\delta,n+1\rangle}$, $f_9^*(a^m_{\langle \alpha,\beta\rangle}) = a^m_{\langle \alpha,\beta\rangle}$, $f_9^*(a^m_{\langle \alpha,\delta,n\rangle}) = a^m_{\langle \alpha,\delta_n\rangle}$.

Let, as above,

$$\begin{aligned}
B_0 &= \langle \{a^m_{\langle 0,0\rangle} \mid m \in \omega\}\rangle,\\
B_1 &= \langle \{a^m_{\langle \alpha,0\rangle} \mid \alpha \in \mu,\ m \in \omega\}\rangle,\\
B_2 &= \langle \{a^m_{\langle 0,\beta\rangle} \mid \beta \in \mu,\ m \in \omega\}\rangle,\\
B_3 &= \langle \{a^m_{\langle \alpha,\beta\rangle} \mid \alpha,\beta \in \mu,\ m \in \omega\}\rangle,\\
B_4 &= \langle \{a^m_{\langle \alpha,\beta\rangle}, a^m_{\langle \alpha,\beta,n\rangle} \mid \alpha,\beta \in \mu,\ n,m \in \omega\}\rangle,\\
B_5 &= \langle \{a^m_{\langle 0,\beta\rangle}, a^m_{\langle 0,\beta,n\rangle} \mid \beta \in \mu,\ n,m \in \omega\}\rangle,\\
B_6 &= \langle \{a^m_{\langle 0,\beta\rangle} \mid \beta \in \mu,\ \mathrm{cf}\beta = \omega,\ m \in \omega\}\rangle,\\
B_7 &= \langle \{a^m_{\langle \alpha,\beta\rangle} \mid \alpha,\beta \in \mu,\ \mathrm{cf}\beta = \omega,\ m \in \omega\}\rangle.
\end{aligned}$$

It is clear that f_3^*, f_4^*, and f_9^* are projections onto B_1, B_2, and B_3, respectively. Let f_{10}^*, f_{11}^*, f_{12}^*, and f_{13}^* be projections onto B_4, B_5, B_6, and B_7, respectively.

All functions which are considered later satisfy the following formula $\varphi^0(f,\dots)$:

$$f_0^* f = f f_0^* = f_0^* \wedge f f_1^* = f_1^* f.$$

Let us see what this formula means. Its first part gives us $f(a_0^m) = ff_0^*(a_0^m) = f_0^*(a_0^m) = a_0^m$ and $f_0^* f(a_i^m) = f_0^*(a_i^m) = a_0^m$, therefore $f(a_i^m) = \alpha_1 a_{i_1}^m + \cdots + \alpha_k a_{i_k}^m$. The second part gives

$$ff_1^*(a_t^m) = f_1^* f(a_t^m) \Leftrightarrow f(a_t^{m-1}) = f_1^* \alpha_1(m) a_{t_1(m)}^m + \cdots + \alpha_k(m) a_{t_k(m)}^m) = \alpha_1(m) a_{t_1(m)}^{m-1} + \cdots + \alpha_k(m) a_{t_k(m)}^{m-1},$$

therefore for every $t \in I^*$ and for all $l, m \in \omega$ we have $\alpha_i(m) = \alpha_i(l)$ and $t_i(m) = t_i(l)$.

Now we apply Lemma 3 with $J = \{\langle \alpha, \beta \rangle \mid \alpha, \beta \in \mu\}$, $J_\beta = \{\langle \alpha, \beta \rangle \mid \alpha \in \mu\}$, $I = I^*$, and $f = f_{14}^*$.

The formula

$$\varphi^1(f, g; f_1^*, \ldots, f_{14}^*) := \varphi^0(f) \wedge \varphi^0(g) \wedge \exists h_1 \, \exists h_2 \, (h_1 f h_1^{-1} = f_2^* \wedge h_2 g h_2^{-1} = f_2^*)$$
$$\wedge \, f_9^* f = f \wedge f_9^* g = g \wedge \exists h \, (ff_{14}^* = f_{14}^* h \wedge hf_9^* = f_9^* h f_9^* \wedge hf = g)$$

says that

1. f and g are conjugate to f_2^*;

2. $\operatorname{Rng} f, \operatorname{Rng} g \subset B_3$;

3. $\exists h \, (h \circ f_{14}^* = f_{14}^* \circ h \wedge \operatorname{Rng} h|_{B_3} \subseteq B_3 \wedge h \circ f = g)$.

We shall write $\varphi^1(f, g; \ldots)$ also in the form $f \leq g$.

If f and g are conjugate to f_2^*, $f(a_{\langle 0,0 \rangle}^m) = a_{\langle \alpha,\beta \rangle}^m$ and $g(a_{\langle 0,0 \rangle}^m) = \tau(a_{\langle \alpha_1,\beta_1 \rangle}^m, \ldots, a_{\langle \alpha_k,\beta_k \rangle}^m)$, then $f \leq g$ if and only if $\beta \leq \beta_1, \ldots, \beta \leq \beta_k$.

The formula

$$\varphi_2(f, f_1^*, \ldots, f_{14}^*) := \varphi^0(f) \wedge (ff_4^* = f_9^* ff_4^*) \wedge (ff_{11}^* = f_{10}^* ff_{11}^*) \wedge (ff_2^* = f_3^* ff_2^*) \wedge (ff_{12}^* = f_{13}^* ff_{12}^*)$$
$$\wedge \, \forall g \, (\varphi^0(g) \wedge \exists h \, (hgh^{-1} = f^*) \wedge f_4^* g = g \Rightarrow \varphi^1(f, fg; f_1^*, \ldots, f_{14}^*))$$
$$\wedge \, (ff_{11}^* f_7^* = f_7^* ff_{11}^*) \wedge (ff_{11}^* f_8^* = f_8^* ff_{11}^*) \wedge (ff_{11}^* f_9^* = f_9^* ff_{11}^*) \wedge (ff_9^* f_3^* = f_3^* ff_9^*)$$

says that

1. $\operatorname{Rng} f|_{B_2} \subseteq B_3$; $\operatorname{Rng} f|_{B_5} \subseteq B_4$; $\operatorname{Rng} f|_{B_0} \subseteq B_1$; $\operatorname{Rng} f|_{B_6} \subseteq B_7$;

2. for any g satisfying $\varphi^0(g)$ and conjugate to f_2^*, if $\operatorname{Rng} g \subseteq B_2$, then $g \leq f \circ g$;

3. $f|_{B_5}$ commutes with f_7^*, f_8^*, and f_9^*;

4. $f|_{B_3}$ commutes with f_3^*.

Then, similarly to Statement 1, we can prove that the formula $\varphi_2(f, \ldots)$ holds in $\operatorname{End}(B')$ if and only if for any $\beta \in \mu$

$$f(a_{\langle 0,\beta \rangle}^m) = \tau(a_{\langle \alpha_1,\beta \rangle}^m, \ldots, a_{\langle \alpha_k,\beta \rangle}^m) \quad \text{and} \quad f(a_{\langle 0,\beta,n \rangle}^m) = \tau(a_{\langle \alpha_1,\beta,n \rangle}^m, \ldots, a_{\langle \alpha_k,\beta,n \rangle}^m)$$

for some linear combination τ and ordinal numbers $\alpha_1, \ldots, \alpha_k \in \mu$ (which do not depend on β).

The formula

$$\varphi^3(f, f_1^*, \ldots, f_{14}^*) := (ff_4^* = f_9^* ff_4^*) \wedge \exists f_1 \, (f_1 f_4^* = ff_4^* \wedge \varphi^2(f_1, f_1^*, \ldots, f_{14}^*))$$

says that

1. $\operatorname{Rng} f|_{B_2} \subseteq B_3$;

2. $\exists f_1 (f_1|_{B_2} = f|_{B_2} \wedge \varphi_2(f_1))$.

The formula $\varphi_3(f,\dots)$ holds if and only if

$$f(a^m_{\langle 0,\beta\rangle}) = \tau(a^m_{\langle\gamma_1,\beta\rangle},\dots,a^m_{\langle\gamma_k,\beta\rangle})$$

for every $\beta\in\mu$ and some $\tau;\gamma_1,\dots,\gamma_k$. This follows immediately from Statement 1.

Let the formula $\varphi_4(f)$ say that

1. $\operatorname{Rng} f|_{B_2}\subseteq B_3$;
2. $\varphi_3(f)$;
3. $\forall g\,(\varphi_3(g)\Rightarrow g\circ f_5\circ f|_{B_0} = f_5\circ f\circ f_5\circ g|_{B_0}\wedge f_5^*\circ f\circ f_5^*\circ f|_{B_0} = f_6^*\circ f|_{B_0}\wedge f_2^*\circ f|_{B_0} = f_2^*|_{B_0}$.

The formula $\varphi_4(f)$ holds if and only if

$$f(a^m_{\langle 0,\beta\rangle}) = \tau(a^m_{\langle\gamma_1,\beta\rangle},\dots,a^m_{\langle\gamma_k,\beta\rangle}),$$

where τ is a beautiful linear combination.

Now we suppose that μ_B is a singular cardinal number.

We let $\mu_1<\mu$, where μ_1 is a regular cardinal and $\mu_1>\omega$. Let $I^*\setminus J = I_0\cup\{\langle\alpha,\delta,n\rangle\mid \alpha\in\mu,\ \delta\in\mu_1,\ \mathrm{cf}\delta=\omega,\ n\in\omega\}$, $|I_0|=\mu$.

For every limit ordinal $\delta\in\mu_1$ such that $\mathrm{cf}\delta=\omega$, similarly to the previous case we shall choose an increasing sequence $(\delta_n)_{n\in\omega}$ of ordinal numbers less than δ, with limit δ, such that for any $\beta\in\mu_1$ and $n\in\omega$ the set $\{\delta\in\mu_1\mid\beta=\delta_n\}$ is a stationary subset on μ_1.

Let

$$\begin{aligned}B_1 &= \langle\{a^m_{\langle\alpha,0\rangle}\mid\alpha\in\mu,\ m\in\omega\}\rangle,\\ B_2 &= \langle\{a^m_{\langle 0,\beta\rangle}\mid\beta\in\mu_1,\ m\in\omega\}\rangle,\\ B_3 &= \langle\{a^m_{\langle\alpha,\beta\rangle}\mid\alpha\in\mu,\ \beta\in\mu_1,\ m\in\omega\}\rangle.\end{aligned}$$

As in the previous case, we can define the functions f_i^* in such a way that for some $\varphi'(\dots)$ the formula $\varphi'(f;\dots)$ holds $\mathrm{End}(B')$ if and only if there exists an ordinal number $\alpha\in\mu$ such that for every $\beta\in\mu_1$ and every $m\in\omega$

$$f(a^m_{\langle 0,\beta\rangle}) = a^m_{\langle\alpha,\beta\rangle}.$$

Let the formula $\varphi^1(f,\dots)$ say that

1. $\operatorname{Rng} f|_{B_0}\subseteq B_2$;
2. for every g we have $\varphi^0(g)\Rightarrow(f\circ g)|_{B_0} = (g\circ f)|_{B_0}$.

It is easy to check that the formula $\varphi^1(f,\dots)$ holds if and only if there exist a linear combination σ and distinct ordinal numbers $\beta_1,\dots,\beta_m\in\mu_1$ such that for every $\alpha\in\mu$ and every $m\in\omega$

$$f(a^m_{\langle\alpha,0\rangle}) = \sigma(a^m_{\langle\alpha,\beta_1\rangle},\dots,a^m_{\langle\alpha,\beta_m\rangle}).$$

As the cardinal number μ_1 is regular, we can use the previous case. Thus, there is a formula $\varphi^2(f;\dots)$ such that $\varphi^2(f)$ holds in $\mathrm{End}(B')$ if and only if there exists $\beta\in\mu_1$ such that for every $\alpha\in\mu$ and every $m\in\omega$

$$f(a^m_{\langle\alpha,0\rangle}) = a^m_{\langle\alpha,\beta\rangle}.$$

Let $\mu=\bigcup\limits_{i\in\mathrm{cf}\mu}\mu_i$, where $\mu_i\in\mu$ and the sequence (μ_i) increases. We have just proved that for every $\gamma\in\mathrm{cf}\mu$ there exists a function $\bar f_\gamma^*$ such that

1. the formula $\varphi^2[f, \bar{f}^*_\gamma]$ holds in $\mathrm{End}(B')$ if and only if there exists $\beta \in \mu^+_\gamma$ such that for all $\alpha \in \mu$ and all $m \in \omega$
$$f(a^m_{\langle\alpha,0\rangle}) = a^m_{\langle\alpha,\beta\rangle};$$
2. $f^*_{\gamma,0}$ is a projection onto
$$\langle\{a^m_{\langle\alpha,\beta\rangle} \mid \alpha \in \mu,\ \beta \in \mu^+_\gamma\}\rangle.$$

Further, there exists a formula φ^3 and a vector of functions g^* such that the formula $\varphi^3(\bar{f}, \bar{g}^*)$ holds if and only if $\bar{f} = \bar{f}^*_\gamma$ for some $\gamma \in \mu$.

Let now the formula $\varphi^4(f, \bar{g}^*)$ say that there exists $\bar{f}_1$ such that $\varphi^3(\bar{f}_1, \bar{g}^*)$ and for every $\bar{f}_2$ satisfying the formulas $\varphi^3(\bar{f}_2, \bar{g}^*)$ and $\mathrm{Rng}(\bar{f}_1)_0 \subseteq \mathrm{Rng}(\bar{f}_2)_0$ we have also $\varphi^2(f, \bar{f}_2)$. If the formula $\varphi^4(f, g^*)$ is true, then there exists $\bar{f}_1 = \bar{f}^*_\gamma$ for some $\gamma \in \mu$ and for every $\bar{f}_2 = f^*_\lambda$ (where $\lambda \geq \gamma$) we have the formula
$$f(a^m_{\langle\alpha,0\rangle}) = a^m_{\langle\alpha,\beta\rangle},$$
where $\beta < \mu^+_\lambda$.

Let f be such that
$$f(a^m_{\langle\alpha,0\rangle}) = a^m_{\langle\alpha,\beta\rangle}, \quad \beta \in \mu.$$
Then $\beta \in \mu^+_\gamma$ for $\gamma \in \mathrm{cf}\mu$ and so the formula $\varphi^4(f, g^*)$ is true for some g^*.

Now we only need to consider the formula $\varphi^4(f^*_5 \circ f \circ f^*_5)$, which is the required formula.

Therefore the case 2 is completely studied, in this case we have (similarly to Sec. 5) a formula (with parameters) which holds for a set of μ_B independent projections f satisfying the formula $\mathrm{Fine}(f)$. Thus we can suppose that we have a set of projections onto μ_B from independent direct summands of the group B isomorphic to the group
$$\bigoplus_{i\in\omega} \mathbb{Z}(p^{n_i}).$$
This set will be denoted by $\mathbf{F}$.

Case 3. $\forall n \in \omega\,(\mu_n < \mu_B)$ and $\mu_B > \omega$. Naturally, in this case the cardinal number μ_B is singular and $\mathrm{cf}\mu_B = \omega$.

Choose in the sequence $(\mu_i)_{i\in\omega}$ an increasing subsequence $(\mu_{n_i})_{i\in\omega}$ with limit μ_B. For every natural i, if the number μ_{n_i} is regular, then by μ^i we shall denote μ_{n_i}, and if it is not regular, then by μ^i we shall denote some regular cardinal number smaller than μ_{n_i} and greater than ω in such a way that in the result the limit of the sequence $(\mu^i)_{i\in\omega}$ is also equal to μ_B. For every natural i suppose that we have a set I^*_i of power μ^i which is the union of the following sets: $J_i = \{\langle\alpha,\beta\rangle \mid \alpha,\beta \in \mu^i\} \cup \{0\}$, $|I^0_i| = \mu^i$, and $I_i = \{\langle\alpha,\delta,n\rangle \mid \alpha \in \mu^i,\ \delta \in \mu^i,\ \mathrm{cf}\delta = \omega,\ n \in \omega\}$. Let us for every $i \in \omega$ have μ^i independent generating direct cyclic summands of order p^{n_i}, denoted by a^i_t, where $t \in I^*$.

Let $\langle\{a^i_t \mid t \in I^*,\ i \in \omega\}\rangle = B'$.

Again we need to change the formulation of Theorem 1, and Lemma 5 will be corrected again: *there exists a formula $\varphi(f)$ with one free variable f such that $\varphi(f)$ holds in* $\mathrm{End}(B')$ *if and only if there exists a sequence of ordinal numbers $(\alpha_i)_{i\in\omega}$, where $\alpha_i \in \mu^i$, such that for every $m \in \omega$ and every $\beta_m \in \mu^m$*
$$f(a^m_{\langle 0,\beta_m\rangle}) = a^m_{\langle\alpha_m,\beta_m\rangle}.$$

All changes in the proof of the theorem with the help of the lemma are clear, so we shall not write them here. In the proof of the lemma we need only the third case. As above, for every limit ordinal $\delta \in \mu^i$ such that $\mathrm{cf}\delta = \omega$, we shall choose an increasing subsequence $(\delta_n)_{n\in\omega}$ of ordinal numbers less than δ, with limit δ, such that for any $\beta \in \mu$ and $n \in \omega$ the set $\{\delta \in \mu \mid \beta = \delta_n\}$ is a stationary subset in μ.

We shall again introduce the functions $f^*_0, \ldots, f^*_{14}$, which will differ a little from the similar functions from the previous case. Namely, let $f^*_1(a^m_t) = a^m_0$, $f^*_2(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle 0,0\rangle}$, $f^*_3(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle\alpha,0\rangle}$, $f^*_4(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle 0,\beta\rangle}$, $f^*_5(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle\beta,\alpha\rangle}$, $f^*_6(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle\alpha,\alpha\rangle}$, for $\delta \in \mu_B$, $\mathrm{cf}\delta = \omega$, $f^*_7(a^m_{\langle\alpha,\delta\rangle}) = a^m_{\langle\alpha,\delta,0\rangle}$, for $\delta \in \mu_B$, $\mathrm{cf}\delta \neq \omega$, $f^*_7(a^m_{\langle\alpha,\delta\rangle}) = a^m_{\langle\alpha,\delta\rangle}$, $f^*_8(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle\alpha,\beta\rangle}$, $f^*_8(a^m_{\langle\alpha,\delta,n\rangle}) = a^m_{\langle\alpha,\delta,n+1\rangle}$, $f^*_9(a^m_{\langle\alpha,\beta\rangle}) = a^m_{\langle\alpha,\beta\rangle}$, $f^*_9(a^m_{\langle\alpha,\delta,n\rangle}) = a^m_{\langle\alpha,\delta_n\rangle}$.

Let, as above,

$$\begin{aligned}
B_0 &= \langle \{a^m_{\langle 0,0\rangle} \mid m \in \omega\}\rangle,\\
B_1 &= \langle \{a^m_{\langle \alpha,0\rangle} \mid \alpha \in \mu^m,\ m \in \omega\}\rangle,\\
B_2 &= \langle \{a^m_{\langle 0,\beta\rangle} \mid \beta \in \mu^m,\ m \in \omega\}\rangle,\\
B_3 &= \langle \{a^m_{\langle \alpha,\beta\rangle} \mid \alpha,\beta \in \mu^m,\ m \in \omega\}\rangle,\\
B_4 &= \langle \{a^m_{\langle \alpha,\beta\rangle}, a^m_{\langle \alpha,\beta,n\rangle} \mid \alpha,\beta \in \mu^m,\ n, m \in \omega\}\rangle,\\
B_5 &= \langle \{a^m_{\langle 0,\beta\rangle}, a^m_{\langle 0,\beta,n\rangle} \mid \beta \in \mu^m,\ n, m \in \omega\}\rangle,\\
B_6 &= \langle \{a^m_{\langle 0,\beta\rangle} \mid \beta \in \mu^m,\ \mathrm{cf}\beta = \omega,\ m \in \omega\}\rangle,\\
B_7 &= \langle \{a^m_{\langle \alpha,\beta\rangle} \mid \alpha,\beta \in \mu^m,\ \mathrm{cf}\beta = \omega,\ m \in \omega\}\rangle.
\end{aligned}$$

Let f^*_{10}, f^*_{11}, f^*_{12}, and f^*_{13} be projections onto B_4, B_5, B_6, and B_7, respectively.

All functions which will be considered later satisfy the following formula $\varphi^0(f,\dots)$:

$$f^*_1 f = f f^*_1 = f^*_1.$$

This formula implies $f(a^m_0) = f f^*_0(a^m_0) = f^*_0(a^m_0) = a^m_0$ and $f^*_0 f(a^m_i) = f^*_0(a^m_i) = a^m_0$, and therefore $f(a^m_i) = \alpha_1 a^m_{i_1} + \dots + \alpha_k a^m_{i_k}$.

For every $m \in \omega$ we apply Lemma 3 with $J^m = \{\langle \alpha,\beta\rangle \mid \alpha,\beta \in \mu^m\}$, $J^m_\beta = \{\langle \alpha,\beta\rangle \mid \alpha \in \mu^m\}$, and $I^m = I^*_m$. Let us for every $m \in \omega$ have the corresponding function f^m on the group $B^m = \langle \{a^m_t \mid t \in I^*_m\}\rangle$. Construct with its help the function f^*_{14}, which coincides with f^m on every subgroup B^m.

As above, the formula

$$\begin{aligned}
f \le g := \varphi^1(f, g; f^*_1, \dots, f^*_{14}) := \varphi^0(f) \wedge \varphi^0(g) \wedge \exists h_1\, \exists h_2\, (h_1 f h_1^{-1} = f^*_2 \wedge h_2 g h_2^{-1} = f^*_2)\\
\wedge f^*_9 f = f \wedge f^*_9 g = g \wedge \exists h\, (f f^*_{14} = f^*_{14} h \wedge h f^*_9 = f^*_9 h f^*_9 \wedge h f = g)
\end{aligned}$$

says that

1. f and g are conjugate to f^*_2;
2. $\mathrm{Rng}\, f, \mathrm{Rng}\, g \subset B_3$;
3. $\exists h\, (h \circ f^*_{14} = f^*_{14} \circ h \wedge \mathrm{Rng}\, h|_{B_3} \subseteq B_3 \wedge h \circ f = g)$.

If f and g are conjugate to f^*_2, $f(a^m_{\langle 0,0\rangle}) = a^m_{\langle \alpha^m,\beta^m\rangle}$, and $g(a^m_{\langle 0,0\rangle}) = \tau^m(a^m_{\langle \alpha^m_1,\beta^m_1\rangle}, \dots, a^m_{\langle \alpha^m_{k_m},\beta^m_{k_m}\rangle})$, then $f \le g$ if and only if $\beta^m \le \beta^m_1, \dots, \beta^m \le \beta^m_{k_m}$.

The formula $\varphi_2(f, f^*_1, \dots, f^*_{14})$ says that

1. $\mathrm{Rng}\, f|_{B_2} \subseteq B_3$; $\mathrm{Rng}\, f|_{B_5} \subseteq B_4$; $\mathrm{Rng}\, f|_{B_0} \subseteq B_1$; $\mathrm{Rng}\, f|_{B_6} \subseteq B_7$;
2. for every g satisfying the formula $\varphi^0(g)$ and conjugate to f^*_2 from $\mathrm{Rng}\, g \subseteq B_2$ it follows that $g \le f \circ g$;
3. $f|_{B_5}$ commutes with f^*_7, f^*_8, and f^*_9;
4. $f|_{B_3}$ commutes with f^*_3.

The formula $\varphi_2(f,\dots)$ holds in $\mathrm{End}(B')$ if and only if for every $m \in \omega$ and every $\beta \in \mu^m$

$$f(a^m_{\langle 0,\beta\rangle}) = \tau^m(a^m_{\langle \alpha^m_1,\beta\rangle}, \dots, a^m_{\langle \alpha^m_{k_m},\beta\rangle}) \ \text{ and }\ f(a^m_{\langle 0,\beta,n\rangle}) = \tau^m(a^m_{\langle \alpha^m_1,\beta,n\rangle}, \dots, a^m_{\langle \alpha^m_{k_m},\beta,n\rangle})$$

for some linear combination τ^m and ordinal numbers $\alpha^m_1, \dots, \alpha^m_{k_m} \in \mu^m$, which do not depends on β.

The formula $\varphi^3(f, f^*_1, \dots, f^*_{14})$ says that

1. $\operatorname{Rng} f|_{B_2} \subseteq B_3$;

2. $\exists f_1 \, (f_1|_{B_2} = f|_{B_2} \wedge \varphi_2(f_1))$.

The formula $\varphi_3(f, \dots)$ holds if and only if for every $m \in \omega$

$$f(a^m_{\langle 0,\beta\rangle}) = \tau^m(a^m_{\langle \gamma_1^m,\beta\rangle}, \dots, a^m_{\langle \gamma_{k_m}^m,\beta\rangle})$$

for every $\beta \in \mu^m$ and some $\tau^m; \gamma_1^m, \dots, \gamma_{k_m}^m$.

The formula $\varphi_4(f)$ says that

1. $\operatorname{Rng} f|_{B_2} \subseteq B_3$;

2. $\varphi_3(f)$;

3. $\forall g \, (\varphi_3(g) \Rightarrow g \circ f_5 \circ f|_{B_0} = f_5 \circ f \circ f_5 \circ g|_{B_0} \wedge f_5^* \circ f \circ f_5^* \circ f|_{B_0} = f_6^* \circ f|_{B_0} \wedge f_2^* \circ f|_{B_0} = f_2^*|_{B_0}$.

The formula $\varphi_4(f)$ holds if and only if for every $m \in \omega$

$$f(a^m_{\langle 0,\beta\rangle}) = \tau^m(a^m_{\langle \gamma_1^m,\beta\rangle}, \dots, a^m_{\langle \gamma_{k_m}^m,\beta\rangle}),$$

where τ is a beautiful linear combination, i.e., there exists a set $\gamma^1, \dots, \gamma^n, \dots$, where $\gamma^i \in \mu^i$ is such that

$$f(a^m_{\langle 0,\beta\rangle}) = a^m_{\langle \gamma^m,\beta\rangle}$$

for all $m \in \omega$ and $\beta \in \mu^m$.

Therefore we suppose that a set of μ_B independent projections satisfying the formula $\mathrm{Fine}(f)$ is fixed. This set will also be denoted by $\mathbf{F}$. In what follows we shall not distinguish the second and the third cases.

7.4 Final Rank of the Basic Subgroup Is Uncountable

Let us change the formula $\mathrm{Fine}(f)$ a little:

$$\begin{aligned}
\mathrm{Fine}(f) := {} & [\forall f' \, (\mathrm{Idem}(f') \wedge f'\varphi_B \neq 0 \Rightarrow f'f \neq 0)] \\
& \wedge [\forall f_1 \forall f_2 \, (\mathrm{Idem}^*(f_1) \wedge \mathrm{Idem}^*(f_2) \wedge o(f_1) = o(f_2) \\
& \wedge \forall c \in Z \, (cf_1 \neq 0 \Rightarrow cf_1 f \neq 0) \wedge \forall c \in Z \, (cf_2 \neq 0 \Rightarrow cf f_2 \neq 0) \Rightarrow f_1 f_2 \neq 0 \wedge f_2 f_1 \neq 0)] \\
& \wedge [\forall \rho' \, (\mathrm{Idem}^*(\rho') \Rightarrow \exists f' \, (\mathrm{Idem}^*(f') \wedge o(f') > o(\rho') \wedge \forall c \in Z \, (cf' \neq 0 \Rightarrow cf'f \neq 0)))] \\
& \wedge [\forall f' \, (\mathrm{Idem}^*(f') \wedge \forall c \in Z \, (cf' \neq 0 \Rightarrow cf' f_B \neq 0) \Rightarrow pf' \neq 0)] \\
& \wedge [\forall \rho' \forall f' \, (\mathrm{Idem}^*(f') \wedge \forall c \in Z \, (cf' \neq 0 \Rightarrow cf' f_B \neq 0) \wedge o(f') = o(\rho') \\
& \Rightarrow \forall f \, (\mathrm{Idem}^*(f) \wedge \forall c \in Z \, (cf \neq 0 \Rightarrow cf f_B \neq 0) \wedge o(f) > o(\rho') \Rightarrow o(f) > o(\rho')^2))].
\end{aligned}$$

The first part of the formula, which is enclosed in square brackets, postulates $fA \subset \varphi_B A = B$. The second part, which is enclosed in square brackets, states that the image of fA contains at most one cyclic direct summand of one order. The third part states that the orders of direct summands of fA are unbounded. The fourth part states that the cyclic summand of fA of the smallest order has order greater than p (i.e., at least not smaller than p^2). Finally, the fifth part states that for every direct cyclic summand in fA of order p^k the next cyclic summand of greater order has order greater than p^{2k}.

We shall again write the formula $\mathrm{Ins}(\psi)$ from the case 1 from Sec. 7.4, which says that

1. for every group fA, where $f \in \mathbf{F}$, there exists a direct cyclic summand of the smallest order such that the action of ψ on it is multiplication by p;

2. for every natural i and every $f \in \mathbf{F}$ there exists a direct cyclic summand $\langle a_i \rangle \subset fA$ of order p^{n_i} such that the action of ψ on it is multiplication by p^i.

Let us fix some endomorphism Ψ that satisfies the formula $\mathrm{Intr}(\Psi)$.

Further, fix an endomorphism $\Gamma\colon B' \to B'$ that for every $f \in \mathbf{F}$ satisfies the following conditions:

1. $f\Gamma f = f\Gamma = \Gamma f$, i.e., the endomorphism Γ maps fA into fA;
2. $\forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \forall \rho'\, (\mathrm{Idem}^*(\rho') \wedge \rho' f = \rho' \Rightarrow o(\rho') \geq o(\rho)) \Rightarrow \Gamma \rho = 0)$, i.e., the endomorphism Γ maps a cyclic summand of the smallest order of fA into zero;
3. $\forall \rho_1\, (\mathrm{Idem}^*(\rho_1) \wedge \rho_1 f = \rho_1 \Rightarrow \exists \rho_2\, (\mathrm{Idem}^*(\rho_2) \wedge \rho_2 f = \rho_2 \wedge o(\rho_1) < o(\rho_2) \wedge \forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \Rightarrow \neg(o(\rho) > o(\rho_1) \wedge o(\rho) < o(\rho_2))) \wedge \rho_1 \Gamma \rho_2 = \Gamma \rho_2 \wedge \forall c \in Z\, (c\rho_1 \neq 0 \Rightarrow c\rho_1 \Gamma \rho_2 \neq 0)))$; this means that Γ maps every generator a_i of a cyclic direct summand of the group fA (isomorphic to $\mathbb{Z}(p^{n_i})$) into the generator a_{i-1} of a cyclic direct summand of fA (isomorphic to $\mathbb{Z}(p^{n_{i-1}})$).

This endomorphism gives us a correspondence between generators of cyclic summands in the group fA, for every $f \in \mathbf{F}$. We shall assume that it is fixed.

At first, suppose for simplicity that the final rank of a basic subgroup of A coincides with its rank, and that it is uncountable. Then $|A| = |B| = \mu$. We suppose that the set $\mathbf{F}$ of μ independent projections on direct summands of the group B, isomorphic to

$$\bigoplus_{i\in\omega} \mathbb{Z}(p^{n_i}),$$

where the sequence (n_i) is such that $n_1 \geq 2$, $n_{i+1} > 2n_i$, is fixed. Let us fix $f \in \mathbf{F}$ and interpret the group A on f.

Let us consider the set End_f of all homomorphisms $h\colon fA \to A$ that satisfy the following condition:

$$\begin{aligned}&\exists \rho (\mathrm{Idem}^*(\rho) \wedge \exists c \in Z\, (\Psi\rho = c\rho) \wedge \rho f = \rho\\ &\quad \wedge \forall \rho'\, (\mathrm{Idem}^*(\rho') \wedge \exists c \in Z\, (\Psi \rho' = c\rho') \wedge \rho' f = \rho' \wedge (o(\rho') < o(\rho) \vee o(\rho') > o(\rho)) \Rightarrow h\rho' = 0)\\ &\quad \wedge \forall c \in Z\, (\Psi\rho = c\rho \Rightarrow pch\rho = 0)).\end{aligned}$$

This condition means that

1. there exists such $i \in \omega$ that $h(a_k) = 0$ for every $k \neq i$;
2. $o(h(a_i)) \leq p^i$.

Naturally, two such homomorphisms h_1 and h_2 are said to be equivalent if $h_1(a_i) = h_2(a_j) \neq 0$ for some $i, j \in \omega$.

Therefore two homomorphisms h_1 and h_2 from End_f are said to be equivalent if there exists a homomorphism h that satisfies the following two conditions:

1. $\forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \exists c \in Z\, (\Psi \rho = c\rho) \wedge h_1 \rho \neq 0 \Rightarrow h\rho = h_1\rho) \wedge \forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \exists c \in Z\, (\Psi\rho = c\rho) \wedge h_2\rho \neq 0 \Rightarrow h\rho = h_2\rho)$; this means that the homomorphism h coincides with h_1 on the element a_i that satisfies $h_1(a_i) \neq 0$, and it coincides with h_2 on the element a_j that satisfies $h_2(a_j) \neq 0$;
2. $\forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \exists c \in Z\, (\Psi\rho = c\rho) \wedge \exists \rho_1\, \exists \rho_2\, (\mathrm{Idem}^*(\rho_1) \wedge \mathrm{Idem}^*(\rho_2) \wedge \rho_1 f = \rho_1 \wedge \rho_2 f = \rho_2 \wedge \exists c_1 \in Z\, (\Psi \rho_1 = c_1\rho_1) \wedge \exists c_2 \in Z\, (\Psi\rho_2 = c_2\rho_2) \wedge h_1\rho_1 \neq 0 \wedge h_2\rho_2 \neq 0 \wedge o(\rho) > o(\rho_1) \wedge o(\rho) \leq o(\rho_2)) \Rightarrow h\Gamma\rho = h\rho)$; this means that $h(a_j) = h(a_{j-1}) = \cdots = h(a_{i+1}) = h(a_i)$.

Thus $h_1(a_i) = h(a_i) = h(a_j) = h_2(a_j)$, which is what we need.

If we factorize the set End_f by this equivalence, we shall obtain the set $\widetilde{\mathrm{End}}_f$. A bijection between this set and the group A is easy to establish. We only need to introduce addition. Namely,

$$\begin{aligned}(h_3 = h_1 \oplus h_2) :=\; &\exists h_1'\, \exists h_2'\, (h_1' \sim h_1 \wedge h_2' \sim h_2\\ &\wedge h_3 = h_1 + h_2 \wedge \forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \forall c \in Z\, (\Psi\rho = c\rho) \Rightarrow (h_1\rho = 0 \Leftrightarrow h_2 \rho = 0))).\end{aligned}$$

We have interpreted the group A for every $f \in \mathbf{F}$, and so we can prove the main theorem for this case.

Proposition 2. *Suppose that p-groups A_1 and A_2 are direct sums $D_1 \oplus G_1$ and $D_1 \oplus G_2$, where the groups D_1 and D_2 are divisible, the groups G_1 and G_2 are reduced and unbounded, $|D_1| \le |G_1|$, $|D_2| \le |G_2|$, B_1 and B_2 are basic subgroups of the groups A_1 and A_2, respectively, and the final ranks of the groups B_1 and B_2 coincide with their ranks and are uncountable. Then elementary equivalence of the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ implies equivalence of the groups A_1 and A_2 in the language $\mathcal{L}_2$.*

Proof. As usual, we shall consider an arbitrary sentence ψ of the second order group language and show an algorithm which translates this sentence ψ to the sentence $\tilde{\psi}$ of the first order ring language such that $\tilde{\psi}$ holds in $\mathrm{End}(A)$ if and only if ψ holds in A.

Consider the formula

$$\begin{aligned}\mathrm{Min}(f) := f \in \mathbf{F} \wedge \forall f' \,(f' \in \mathbf{F} \Rightarrow \forall c \forall \rho' \forall \rho \,(c \in Z \wedge \mathrm{Idem}^*(\rho') \wedge \mathrm{Idem}^*(\rho) \\ \wedge\, \rho f = \rho \wedge \rho' f' = \rho' \wedge \Psi \rho' = c\rho' \wedge \Psi\rho = c\rho \Rightarrow o(\rho) \le o(\rho'))).\end{aligned}$$

This formula gives us a summand $f(A)$ in B such that for every $i \in \omega$ the number $n_i(f)$ is minimal among all $n_i(f')$ for $f' \in \mathbf{F}$.

Consider the formula

$$\begin{aligned}\mathrm{Basic}(\Lambda) := \exists f \,(f \in \mathbf{F} \wedge \mathrm{Min}(f) \wedge \forall f' \forall c \forall \rho' \,(f' \in \mathbf{F} \wedge c \in Z \wedge \mathrm{Idem}^*(\rho') \wedge \rho' f' = \rho' \\ \wedge\, \Psi\rho' = c\rho' \Rightarrow \exists \rho \,(\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \Psi \rho = c\rho \wedge \rho\Lambda\rho' = \Lambda\rho' \wedge \forall c' \in Z \,(c\rho \ne 0 \Rightarrow c\rho\Lambda\rho' \ne 0)))).\end{aligned}$$

This formula defines an endomorphism Λ that maps a_t^i to a_0^i for every $i \in \omega$ and every $t \in \mu$. The corresponding f_0 will be denoted by $f_{\min}$.

Translate the sentence ψ to the sentence

$$\tilde{\psi} := \exists \bar{g}\, \exists \Gamma\, \exists \Psi\, \exists \Lambda\, \exists f_{\min} \in \mathbf{F} \psi'(\bar{g}, \Gamma, \Psi, \Lambda, f_{\min}),$$

where the formula $\psi'(\dots)$ is obtained from the sentence ψ with the help of the following translations of subformulas from ψ:

1. the subformula $\forall x$ is translated to the subformula $\forall x \in \widetilde{\mathrm{End}}_{f_{\min}}$;
2. the subformula $\exists x$ is translated to the subformula $\exists x \in \widetilde{\mathrm{End}}_{f_{\min}}$;
3. the subformula $\forall P_m(v_1, \dots, v_m)(\dots)$ is translated to the subformula
$$\forall f_1^P \dots \forall f_m^P \left(\forall g \in \mathbf{F} \left(\bigwedge_{i=1}^m (f_i^P g \in \mathrm{End}_g) \right) \Rightarrow \dots \right);$$
4. the subformula $\exists P_m(v_1, \dots, v_m)(\dots)$ is translated to the subformula
$$\exists f_1^P \dots \exists f_m^P \left(\forall g \in \mathbf{F} \left(\bigwedge_{i=1}^m (f_i^P g \in \mathrm{End}_g) \right) \wedge \dots \right);$$
5. the subformula $x_1 = x_2$ is translated to the subformula $x_1 \sim x_2$;
6. the subformula $x_1 = x_2 + x_3$ is translated to the subformula $x_1 \sim x_2 \oplus x_3$;
7. the subformula $P_m(x_1, \dots, x_m)$ is translated to the subformula
$$\exists g \in \mathbf{F} \left(\bigwedge_{i=1}^m (f_i^P g) = x_i \Lambda g \right).$$

The rest of the proof is similar to the previous cases. $\square$

Now we shall consider the case where the final rank of B is greater than ω and does not coincide with the rank of B.

In this case, $A = G \oplus G'$, where the group G satisfies the conditions of the previous proposition, the group G' is bounded, and its power is greater than $|G|$. Let $|G| = \mu$ and $|A| = |G'| = \mu'$.

Let us define, for the group G, the set $\mathbf{F}$ from Proposition 2, and the set $\mathbf{F}'$ of μ' independent projections on countably generated subgroups of the group G', from Sec. 5.

The formula

$$\mathrm{Add}(\varphi) := \forall f' \in \mathbf{F}' \, \exists f \in \mathbf{F} \, \forall \rho' \, (\mathrm{Idem}^*(\rho') \wedge \rho' f' = \rho' \Rightarrow \exists \rho \, (\mathrm{Idem}^*(\rho) \wedge \rho f = \rho \wedge \varphi \rho' = \rho \varphi \rho' \neq 0))$$

defines a functions from the set $\mathbf{F}'$ onto the set $\mathbf{F}$.

Proposition 3. *Suppose that p-groups A_1 and A_2 are direct sums $D_1 \oplus G_1$ and $D_1 \oplus G_2$, where the groups D_1 and D_2 are divisible, the groups G_1 and G_2 are reduced and unbounded, $|D_1| < |G_1|$, $|D_2| < |G_2|$, B_1 and B_2 are basic subgroups of the groups A_1 and A_2, respectively, and the final ranks of the groups B_1 and B_2 do not coincide with their ranks and are uncountable. Then elementary equivalence of the rings* $\mathrm{End}(A_1)$ *and* $\mathrm{End}(A_2)$ *implies equivalence of the groups A_1 and A_2 in the language $\mathcal{L}_2$.*

Proof. We only write an algorithm of translation. Let us translate the sentence ψ to the sentence

$$\tilde{\psi} := \exists \bar{g} \, \exists \Gamma \, \exists \Psi \, \exists \Lambda \, \exists f_{\min} \in \mathbf{F} \, \exists \bar{g}_1 \ldots \exists \bar{g}_k \, \exists \bar{g}' \, \exists \tilde{g} \in \mathbf{F}' \psi'(\bar{g}, \Gamma, \Psi, \Lambda, f_{\min}),$$

where the formula $\psi'(\ldots)$ is obtained from the sentence ψ with the help of the following translations of subformulas from ψ:

1. the subformula $\forall x$ is translated to the subformula $\forall x \in \widetilde{\mathrm{End}}_{f_{\min}} \forall x' \in \widetilde{\mathrm{End}}_{\tilde{g}}$;
2. the subformula $\exists x$ is translated to the subformula $\exists x \in \widetilde{\mathrm{End}}_{f_{\min}} \exists x' \in \widetilde{\mathrm{End}}_{\tilde{g}}$;
3. the subformula $\forall P_m(v_1, \ldots, v_m)(\ldots)$ is translated to the subformula

$$\forall f_1^P \ldots \forall f_m^P \, \forall f_1^{P'} \ldots \forall f_m^{P'} \, \forall \varphi_1^P \ldots \forall \varphi_m^P \left(\bigwedge_{i=1}^m \mathrm{Add}(\varphi_i^P) \right. \\ \left. \wedge \, \forall g \in \mathbf{F} \left(\bigwedge_{i=1}^m f_i^P g \in \mathrm{End}_g \right) \wedge \forall g \in \mathbf{F}' \left(\bigwedge_{i=1}^m f_i^{P'} g \in \mathrm{End}_g \right) \Rightarrow \ldots \right);$$

4. the subformula $\exists P_m(v_1, \ldots, v_m)(\ldots)$ is translated to the subformula

$$\exists f_1^P \ldots \exists f_m^P \, \exists f_1^{P'} \ldots \exists f_m^{P'} \, \exists \varphi_1^P \ldots \exists \varphi_m^P \left(\bigwedge_{i=1}^m \mathrm{Add}(\varphi_i^P) \right. \\ \left. \wedge \, \forall g \in \mathbf{F} \left(\bigwedge_{i=1}^m f_i^P g \in \mathrm{End}_g \right) \wedge \forall g \in \mathbf{F}' \left(\bigwedge_{i=1}^m f_i^{P'} g \in \mathrm{End}_g \right) \wedge \ldots \right);$$

5. the subformula $x_1 = x_2$ is translated to the subformula $x_1 \sim x_2 \wedge x_1' \sim x_2'$;
6. the subformula $x_1 = x_2 + x_3$ is translated to the subformula $x_1 \sim x_2 \oplus x_3 \wedge x_1' \sim x_2' \oplus x_3'$;
7. the subformula $P_m(x_1, \ldots, x_m)$ is translated to the subformula

$$\exists g \in \mathbf{F} \, \exists g' \in \mathbf{F}' \left(\bigwedge_{i=1}^m (\varphi_i^P(g') = g \wedge f_i^P g = x_i \Lambda g \wedge f_i^{P'} g' = x_i' \tilde{g} h) \right). \quad \square$$

7.5 The Countable Restriction of the Second Order Theory of the Group in the Case Where the Rank of the Basic Subgroup Is Countable

Let the group A have a countable basic subgroup B. As above, we suppose that an endomorphism φ_B with the image B is fixed.

Further, we suppose that an endomorphism f_B considered in the case 1 in Sec. 7.3 and an endomorphism Ψ satisfying the formula $\mathrm{Ins}(\Psi)$ from the same section are fixed. As we remember,

$$B = \varphi_B(A) \supset B' = f_B(A) \cong \bigoplus_{i\in\omega} \mathbb{Z}(p^{n_i}),$$

where $n_0 \geq 2$, $n_{i+1} > 2n_i$. Generators of cyclic summands of $f_B(A)$, where $\Psi(a_i) = p^i a_i$, will be denoted by a_i ($i \in \omega$).

Further, as in Sec. 7.4, we fix a homomorphism $\Gamma\colon B' \to B'$ satisfying the conditions

1. $\forall f\, (\mathrm{Idem}^*(f) \wedge \forall c \in Z\, (cf \neq 0 \Rightarrow cff_B \neq 0) \Rightarrow f\Gamma f = f\Gamma = \Gamma f)$, i.e., $f \in \mathrm{End}(B')$;
2. $\forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \forall c \in Z\, (c\rho \neq 0 \Rightarrow c\rho f_B \neq 0) \wedge \forall \rho'\, (\mathrm{Idem}^*(\rho') \wedge \forall c \in Z\, (c\rho' \neq 0 \Rightarrow c\rho' f_B \neq 0) \Rightarrow o(\rho') \geq o(\rho)) \Rightarrow \Gamma\rho = 0)$, i.e., the endomorphism Γ maps a cyclic summand of the smallest order (in the group) into 0;
3. $\forall \rho_1\, (\mathrm{Idem}^*(\rho_1) \wedge \forall c \in Z\, (\rho_1 \neq 0 \Rightarrow c\rho_1 f_B \neq 0) \Rightarrow \exists \rho_2\, (\mathrm{Idem}^*(\rho_2) \wedge \forall c \in Z\, (c\rho_2 \neq 0 \Rightarrow c\rho_2 f_B \neq 0) \wedge o(\rho_1) < o(\rho_2) \wedge \forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \forall c \in Z\, (c\rho \neq 0 \Rightarrow c\rho f_B \neq 0) \Rightarrow \neg(o(\rho) > o(\rho_2) \wedge o(\rho) < o(\rho_2))) \wedge \rho_1\Gamma = \Gamma\rho_2 = \rho_1\Gamma\rho_2 \wedge \forall c \in Z\, (c\rho_1 \neq 0 \Rightarrow c\rho_1\Gamma\rho_2 \neq 0)))$, i.e., Γ maps every generator a_i of a cyclic direct summand of the group B' (isomorphic to $\mathbb{Z}(p^{n_i})$) into the generator a_{i-1} of a cyclic direct summand (isomorphic to $\mathbb{Z}(p^{n_{i-1}})$).

It is clear that for interpretation of the group A for our function f_B we can use the sets End_{f_B} and $\widetilde{\mathrm{End}}_{f_B}$ from the previous subsection. Therefore, every element $a \in A$ is mapped to a class of homomorphisms $g\colon B' \to A$ satisfying the conditions $g(a_i) = a$ and $g(a_j) = 0$ if $j \neq i$, where i is such that $p^i \geq o(a)$.

Now suppose that we want to interpret some set $X = \{x_i\}_{i\in I} \subset A$, where $|X| \leq \omega$, in the ring $\mathrm{End}(A)$. It is clear that there exists a sequence (k_i), $i \in I$, and a homomorphism $h\colon B' \to A$ such that

$$h(a_{k_i})x_i, \quad i \in I,$$

and $o(x_i) \leq p^{k_i}$. The set $\{x_i\}$ is uniquely defined by the homomorphism h and the sequence (k_i). Therefore, every set $\{x_i\}$ can be mapped to a pair of endomorphisms consisting of a projection onto the subgroup $\langle\{a_{k_i} \mid i \in I\}\rangle$ and a homomorphism h.

Similarly, every n-place relation in A is mapped to a projection on $\langle\{a_{k_i} \mid i \in I\}\rangle$ and an n-tuple of homomorphisms $h_1, \ldots, h_n$.

Introduce the formulas

$$\begin{aligned}\mathrm{Proj}(\rho) := {}& \forall \rho'\, (\mathrm{Idem}^*(\rho') \wedge \forall c \in Z\, (c\rho' \neq 0 \Rightarrow c\rho'\rho \neq 0) \\ & \Rightarrow \forall c \in Z\, (c\rho' \neq 0 \Rightarrow c\rho' f_B \neq 0) \wedge \exists \rho''\, (\mathrm{Idem}^*(\rho'') \wedge \forall c \in Z\, (c\rho'' \neq 0 \Rightarrow c\rho''\rho \neq 0) \\ & \wedge o(\rho'') = o(\rho') \wedge \exists c \in Z\, (\Psi\rho'' = c\rho'')) \wedge \exists c \in Z\, (\Psi\rho' = c\rho'))\end{aligned}$$

(a projection on a direct summand in B', generated by $\{a_{k_i} \mid i \in I\}$) and

$$\mathrm{Hom}(h) := \forall \rho'\, (\mathrm{Idem}^*(\rho') \wedge \forall c \in Z\, (c\rho' \neq 0 \Rightarrow c\rho' f_B \neq 0) \Rightarrow \exists c \in Z\, (\Psi\rho' = c\rho' \Rightarrow pch\rho' = 0)).$$

Now we are ready to prove the following proposition.

Proposition 4. *Let p-groups A_1 and A_2 be unbounded and have countable basic subgroups. Then* $\mathrm{End}(A_1) \equiv \mathrm{End}(A_2)$ *implies* $\mathrm{Th}_2^\omega(A_1) = \mathrm{Th}_2^\omega(A_2)$.

Proof. Suppose that we have a sentence $\psi \in \mathrm{Th}_2^{\omega}(A_1)$. Then for every predicate variable $P_n(v_1, \dots, v_n)$ included in ψ, the set $\{\langle a_1, \dots, a_n \rangle \in A^n \mid P(a_1, \dots, a_n)\}$ is at most countable.

We shall show a translation of the sentence ψ into the first order sentence $\tilde{\psi} \in \mathrm{Th}_1(\mathrm{End}(A_1))$.

Let us translate the sentence ψ to the sentence

$$\tilde{\psi} = \exists \varphi_B \, \exists f_B \, \exists \Psi \, \exists \Gamma \, \psi'(\varphi_B, f_B, \Psi, \Gamma),$$

where the formula $\psi'(\dots)$ is obtained from the sentence ψ with the help of the following translations of subformulas from ψ:

1. the subformula $\forall x$ is translated to the subformula $\forall x \in \widetilde{\mathrm{End}}_{f_B}$;
2. the subformula $\exists x$ is translated to the subformula $\exists x \in \widetilde{\mathrm{End}}_{f_B}$;
3. the subformula $\forall P_m(v_1, \dots, v_m)(\dots)$ is translated to the subformula
$$\forall \rho^P \, \forall h_1^P \dots \forall h_m^P \, (\mathrm{Proj}(\rho^P) \wedge \mathrm{Hom}(h_1^P) \wedge \dots \wedge \mathrm{Hom}(h_m^P) \Rightarrow \dots);$$
4. the subformula $\exists P_m(v_1, \dots, v_m)(\dots)$ is translated to the subformula
$$\exists \rho^P \, \exists h_1^P \dots \exists h_m^P \, (\mathrm{Proj}(\rho^P) \wedge \mathrm{Hom}(h_1^P) \wedge \dots \wedge \mathrm{Hom}(h_m^P) \wedge \dots);$$
5. the subformula $x_1 = x_2$ is translated to the subformula $x_1 \sim x_2$;
6. the subformula $x_1 = x_2 + x_3$ is translated to the subformula $x_1 \sim x_2 \oplus x_3$;
7. the subformula $P_m(x_1, \dots, x_m)$ is translated to the subformula
$$\exists \rho \, (\mathrm{Idem}^*(\rho) \wedge \rho \rho^P = \rho \wedge \exists c \in Z \, (\Psi \rho = c \rho) \wedge h_1^P \rho = x_1 \wedge \dots \wedge h_m^P \rho = x_m),$$
i.e., there exists a cyclic summand $\langle a_{k_i} \rangle$ in B' such that $i \in I$ and $h_1(a_{k_i}) = x_1, \dots, h_m(a_{k_i}) = x_m$.

□

7.6 The Final Rank of the Basic Subgroup Is Equal to ω and Does Not Coincide with Its Rank

As above, we suppose that $A = D \oplus G$, where the group D is divisible, the group G is reduced and unbounded, and $|D| < |G|$. Let a basic subgroup $\bar{B}$ of A (and of G) have the form $B \oplus B'$, where $|B'| = |\bar{B}|$, $|B| = \omega$, and B' is bounded.

Note that from $|D| < |G|$ it follows that $|D| \leq \omega$, because we assume the continuum-hypothesis, which implies $|B'| = |\bar{B}| = \omega_1 = 2^{\omega} = c$.

The condition $|D| \leq \omega$ means that if the groups $A_1 = D_1 \oplus G_1$ and $A_2 = D_2 \oplus G_2$ have the described type and $\mathrm{End}(A_1) \equiv \mathrm{End}(A_2)$, then $D_1 \cong D_2$, and in order to prove $A_1 \equiv_{\mathcal{L}_2} A_2$ we only need to prove $G_1 \equiv_{\mathcal{L}_2} G_2$ (with the condition that an endomorphism between D and G is fixed, see Proposition 3).

Therefore for simplicity of arguments we suppose that the group A is reduced, i.e., $A = G$ and $D = 0$.

We fix an endomorphism φ_B with the image B. Further, let us fix an endomorphism f_B with the image B' which is a direct summand in B isomorphic to

$$\bigoplus_{i \in \omega} \mathbb{Z}(p^{n_i}),$$

where $n_0 \geq 2$, $n_{i+1} > 2n_i$.

Naturally, we also suppose that endomorphisms Ψ and Γ, described in Secs. 7.3 and 7.5, are fixed.

Let ρ_1 and ρ_2 be indecomposable projections onto cyclic direct summands of the group B' satisfying the formulas $\exists c \in Z\,(\Psi\rho_1 = c\rho_1) \wedge \exists c \in Z\,(\Psi\rho_2 = c\rho_2)$ and $o(\rho_1) > o(\rho_2)$. Then we shall write $\gamma \in \langle\Gamma\rangle_{\rho_1,\rho_2}$ if an endomorphism γ satisfies the formula

$$\begin{aligned}&\exists\gamma'\,(\gamma'\rho_2 = \rho_2 \wedge \forall\rho\,(\mathrm{Idem}^*(\rho) \wedge \forall c \in Z\,(c\rho \neq 0 \Rightarrow c\rho f_B \neq 0)\\ &\qquad \wedge \exists c \in Z\,(\Psi\rho = c\rho) \wedge o(\rho) \leq o(\rho_1) \wedge o(\rho) > o(\rho_2) \Rightarrow \gamma'\rho = \gamma'\Gamma\rho) \wedge \gamma\rho_1 = \gamma'\rho_1).\end{aligned}$$

This formula means that there exists an endomorphism $\gamma' \colon B' \to B'$ such that, if a_i is a generator in $\rho_2 A$ and a_{i+k} is a generator in $\rho_1 A$, then we have

1. $\gamma'(a_i) = a_i$;
2. $\forall i \in \{1, \dots, k\}\, \gamma'(a_{i+k}) = \gamma'(a_{i+k-1}) = \cdots = \gamma'(a_{i+1}) = \gamma'(a_i) = a_i$.

Further, $\gamma(a_{i+k}) = \gamma'(a_{i+k}) = a_i$. Thus, we have $\gamma(a_{i+k}) = a_i$.

Now consider the formula

$$\begin{aligned}\mathrm{Onto}(\Lambda) := \Bigg[&\forall\bar\rho\,\forall\bar c\,\Bigg(\mathrm{Idem}^*(\bar\rho) \wedge \forall c \in Z\,(c\bar\rho \neq 0 \Rightarrow c\bar\rho\varphi_B \neq 0)\\
&\wedge \bar c \in Z \wedge \bar c\bar\rho \neq 0 \Rightarrow \exists\rho_1 \dots \exists\rho_{p-1}\Bigg(\Bigg(\bigwedge_{i,j=1}^{p-1} \rho_i\rho_j = \rho_j\rho_i = 0\Bigg) \wedge \mathrm{Idem}^*(\rho_1) \wedge \cdots \wedge \mathrm{Idem}^*(\rho_{p-1})\\
&\wedge \Bigg(\bigwedge_{i=1}^{p-1} \forall c \in Z\,(c\rho_i \neq 0 \Rightarrow c\rho_i f_B \neq 0)\Bigg) \wedge \Bigg(\bigwedge_{i=1}^{p-1} \exists c \in Z\,(\Psi\rho_i = c\rho_i)\Bigg)\\
&\wedge \Bigg(\bigwedge_{i=1}^{p-1} \bar c\bar\rho\Lambda\rho_i \neq 0\Bigg) \wedge \Bigg(\bigwedge_{i=1}^{p-1} \forall c \in Z\,(c\bar c\bar\rho \neq 0 \Rightarrow c\bar c\bar\rho\Lambda\rho_i \neq 0)\Bigg)\\
&\wedge \Bigg(\bigwedge_{i,j=1;\ i\neq j}^{p-1} o(\rho_i) > o(\rho_j) \Rightarrow \forall\gamma \in \langle\Gamma\rangle_{\rho_i,\rho_j}(\Lambda\gamma\rho_i \neq \Lambda\rho_i)\Bigg)\Bigg)\Bigg)\Bigg]\\
&\wedge [\mathrm{Hom}(\Lambda)] \wedge [\forall\rho_1\,\forall\rho_2\,(\mathrm{Idem}^*(\rho_1) \wedge \mathrm{Idem}^*(\rho_2)\\
&\wedge \forall c \in Z\,(c\rho_1 \neq 0 \Rightarrow c\rho_1 f_B \neq 0) \wedge \forall c \in Z\,(c\rho_2 \neq 0 \Rightarrow c\rho_2 f_B \neq 0)\\
&\wedge \exists c \in Z\,(\Psi\rho_1 = c\rho_1) \wedge \exists c \in Z\,(\Psi\rho_2 = c\rho_2) \wedge o(\rho_1) \geq o(\rho_2)\\
&\Rightarrow \exists\rho_3(\mathrm{Idem}^*(\rho_3) \wedge \forall c \in Z\,(c\rho_3 \neq 0 \Rightarrow c\rho_3 f_B \neq 0) \wedge \exists c \in Z\,(\Psi\rho_3 = c\rho_3)\\
&\wedge ((o(\rho_3) > o(\rho_1) \wedge o(\rho_3) > o(\rho_2) \wedge \exists\gamma_1 \in \langle\Gamma\rangle_{\rho_3,\rho_1}\,\exists\gamma_2 \in \langle\Gamma\rangle_{\rho_3,\rho_2}\,(\Lambda\rho_3 = \Lambda\gamma_1\rho_3 + \Lambda\gamma_2\rho_3))\\
&\vee (o(\rho_3) < o(\rho_1) \wedge o(\rho_3) < o(\rho_2) \wedge \exists\gamma_2 \in \langle\Gamma\rangle_{\rho_1,\rho_2}\,\exists\gamma_3 \in \langle\Gamma\rangle_{\rho_1,\rho_3}\,(\Lambda\gamma_3\rho_1 = \Lambda\rho_1 + \Lambda\gamma_2\rho_1))\\
&\vee (o(\rho_3) < o(\rho_1) \wedge o(\rho_3) > o(\rho_2) \wedge \exists\gamma_2 \in \langle\Gamma\rangle_{\rho_1,\rho_2}\,\exists\gamma_3 \in \langle\Gamma\rangle_{\rho_1,\rho_3}\,(\Lambda\gamma_3\rho_1 = \Lambda\gamma_1 + \Lambda\gamma_2\rho_1)))))]\\
&\wedge [\forall\rho_1\rho_2(\mathrm{Idem}^*(\rho_1) \wedge \mathrm{Idem}^*(\rho_2) \wedge \forall c \in Z\,(c\rho_1 \neq 0 \Rightarrow c\rho_1 f_B \neq 0)\\
&\wedge \forall c \in Z\,(c\rho_2 \neq 0 \Rightarrow c\rho_2 f_B \neq 0) \wedge \exists c \in Z\,(\Psi\rho_1 = c\rho_1) \wedge \exists c \in Z\,(\Psi\rho_2 = c\rho_2)\\
&\wedge o(\rho_1) > o(\rho_2) \Rightarrow \exists\rho\,(\mathrm{Idem}^*(\rho) \wedge \forall c \in Z\,(c\rho \neq 0 \Rightarrow c\rho\varphi_B \neq 0) \wedge \rho\Lambda\rho_2 \neq 0)\\
&\wedge \forall\gamma \in \langle\Gamma\rangle_{\rho_1,\rho_2}\,(\Lambda\rho_1 \neq \Lambda\gamma\rho_1))].\end{aligned}$$

The first condition in brackets means that for every generator b of the cyclic direct summand $\langle b\rangle$ of the group B and for every integer p-adic number c there exist at least $p-1$ numbers $i_1, \dots, i_{p-1} \in \omega$ such that $\Lambda(a_{i_k}) = \xi_k c \cdot b$, where ξ_k are distinct and not divisible by p, $\xi_k c \cdot b \neq \xi_l c \cdot b$ for all $k \neq l$.

The following conditions mean that for any a_i and a_j there exists a_k such that $\Lambda(a_k) = \Lambda(a_)\, + \Lambda(a_j)$. Therefore the endomorphism Λ is an epimorphism $B' \to B$.

The last condition in brackets means that Λ induces a bijection between the set $\{a_i \mid i \in \omega\}$ and the group B, with the condition $o(\Lambda(a_i)) \leq p^i$.

Let us fix an endomorphism Λ.

Now recall that we have a bounded group B' of some uncountable power μ that has a definable set $\mathbf{F} = \mathbf{F}(\bar{g})$ consisting of μ independent indecomposable projections onto direct summands of the group B', and a set $\mathbf{F}' = \mathbf{F}'(\bar{g}')$ consisting of μ independent projections onto countably generated direct summands of the group B', and for every $f \in \mathbf{F}'$ the set consisting of $f_t \in \mathbf{F}$ such that $f_t A$ is a direct summand in fA is countable. Denote the subset $\{f_t \in \mathbf{F} \mid f_t A \subset fA\}$ of $\mathbf{F}$ by $\mathbf{F}_f$. It is clear that the set $\mathbf{F}_f$ is definable.

Let us fix endomorphisms Π_1 and Π_2 introducing the order on the set $f(A)$, where $f \in \mathbf{F}$:

$$\begin{aligned}
&\mathrm{Order}(\Pi_1, \Pi_2) := \forall f \in \mathbf{F}\, (\exists! f_0 \in \mathbf{F}_f\, (\Pi_2 f_0 = 0 \wedge \\
&\quad \wedge \forall f_1\, (f_1 \in \mathbf{F}_f \wedge f_1 \neq f_0 \Rightarrow \exists f_2 \in \mathbf{F}_f\, (f_1 \neq f_2 \wedge \\
&\quad \wedge f_2 \Pi_2 = \Pi_2 f_1 = f_2 \Pi_2 f_1 \wedge \forall c \in Z\, (c f_1 \neq 0 \Rightarrow c \Pi_2 f_1 \neq 0) \wedge \\
&\quad \wedge f_1 \Pi_1 = \Pi_1 f_2 = f_1 \Pi_1 f_2 \wedge \forall c \in Z\, (c f_2 \neq 0 \Rightarrow c \Pi_1 f_2 \neq 0)) \wedge \\
&\quad \wedge \forall f_1\, \forall f_2 \in \mathbf{F}_f \Bigg(f_1 \neq f_2 \Rightarrow \bigwedge_{i=1}^{2} \forall f_3, f_4 \in \mathbf{F}_f \\
&\quad (f_3 \Pi_i f_1 = f_3 \Pi_i = \Pi f_1 \neq 0 \wedge f_4 \Pi_i f_2 = f_4 \Pi_i f_2 = f_4 \Pi_i = \Pi_i f_2 \neq 0 \Rightarrow f_3 \neq f_4) \Bigg) \wedge \\
&\quad \wedge \forall f_1 \in \mathbf{F}_f\, \exists f_2 \in \mathbf{F}_f\, (\Pi_2 f_2 = f_1 \Pi_2 = f_1 \Pi_2 f_2 \neq 0) \wedge \forall f_1 \in \mathbf{F}_f\, (\Pi_2 \Pi_1 f = f \wedge \\
&\quad \wedge \forall f'\, (\mathrm{Idem}^*(f') \wedge f' f = f' \wedge \mathrm{Fin}(f') \wedge f' \neq 0 \Rightarrow \\
&\quad \Rightarrow \exists f_1, f_2 \in \mathbf{F}_f\, (f_1 f = f_1 \wedge f_2 f = f_2 \wedge f_2 \Pi_2 = f_2 \Pi_2 f_1 = \Pi_2 f_1 \neq 0)))))).
\end{aligned}$$

Up to the order Π_1 we can suppose that for every $f_t \in \mathbf{F}$ ($t \in \mu$) we have a basis in $f_t(A)$ consisting of $f_t^0, f_t^1, \ldots$, where $\Pi_1(f_t^i A) = f_t^{i+1} A$.

Now let us write the conditions for the homomorphism Δ.

1. For every function $f \in \mathbf{F}$ we shall introduce (by a formula) a function

$$\rho_{\mathrm{on},f} \colon B' \to f_t A$$

such that

$$\rho_{\mathrm{on},f}(a_i) = f_t^i$$

for all natural i.

The condition for the homomorphism Δ now has the form

$$\forall f \in \mathbf{F}\, \exists \beta\, (\mathrm{Hom}(\beta) \wedge (1) \wedge (2) \wedge (3) \wedge (4)),$$

where

(1) $\forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \forall c \in Z\, (c\rho \neq 0 \Rightarrow c\rho f_B \neq 0) \wedge \exists c \in Z\, (\Psi\rho = c\rho) \Rightarrow \forall c \in Z\, (\Psi\rho = c\rho \Rightarrow c\beta\rho \neq 0 \wedge pc\beta\rho \neq 0))$, and this means that $o(\beta(a_i)) = p^i$;

(2) $\forall \rho \forall \rho'\, (\mathrm{Idem}^*(\rho) \wedge \mathrm{Idem}^*(\rho') \wedge \forall c \in Z\, (c\rho \neq 0 \Rightarrow c\rho f_B \neq 0) \wedge \forall c \in Z\, (c\rho' \neq 0 \Rightarrow c\rho' \varphi_B \neq 0)$ $\exists c \in Z\, (\Psi\rho = c\rho) \Rightarrow \neg(\rho' \beta\rho = \beta\rho))$, and this means that $\beta(a_i) \notin B$;

(3) $\forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \forall c \in Z\, (c\rho \neq 0 \Rightarrow c\rho f_B \neq 0) \wedge \exists c \in Z\, (\Psi\rho = c\rho) \Rightarrow p\beta\rho = p\beta(\Gamma\rho) + \Lambda\Delta\rho_{\mathrm{on},f}\rho)$, and this means that $p\beta(a_i) = \rho(a_{i-1}) + \Lambda\Delta(f_t^i)$;

(4) $\forall c \in Z\, (c\Lambda\Delta\rho_{\mathrm{on},f}\rho \neq 0 \Rightarrow c\beta\rho \neq 0)$, and this means that $o(\Lambda\Delta(f_t^i)) \leq p^i$.

This condition means that for every f_t, where $t \in \mu$, Δ is a mapping from $\{f_t^i \mid i \in \omega\}$ into $\{a_i \mid i \in \omega\}$ such that there is a sequence $c_{1,t}, \ldots, c_{m,t}, \cdots \in A$ such that $c_{i,t} \notin B$, $o(c_{i,t}) = p^i$, $pc_{i,t} = c_{i-1,t} + b_{i,t}$, where $b_{i,t} \in B$, $b_{i,t} = \beta(\Delta(f_t^i))$, and $o(b_{i,t}) \leq p^i$.

As we know, such a sequence can be considered as a sequence of the elements of a quasibasis of A, and it can be assumed to be uniquely defined with the help of the sequence

$$(b_{i,t} \mid i \in \omega, b_{i,t} = \beta(\Delta(f_t^i A))).$$

An endomorphism β is uniquely defined by every $f \in \mathbf{F}$, for abbreviation we shall write $\mathrm{Quasi}_f(\beta)$.

2. Every set of elements from Γ gives us a set of sequences of elements of the quasibasis

$$\{f_t \mid t \in J\} \leftrightarrow \{(c_{1,t}, \dots, c_{i,t}, \dots) \mid t \in J\} = C_J,$$

and the set C_J gives us a linear space $\bar{C}_J = \langle C_J \rangle$ such that we can determine whether some sequence belongs to this linear space or not. We do not want to write the formulas, because they are too complicated, but we shall write only conditions: (a) every existing sequence belongs to $\bar{C}_I$; (b) linear spaces $\langle C_{J_1} \rangle$ and $\langle C_{J_2} \rangle$ $(J_1 \cap J_2 = \varnothing)$ intersect by B.

Therefore, our homomorphism Δ is a bijection between the set $\{f_t^i \mid t \in I\}$ and the quasibasis $\{c_{it} \mid i \in \omega,\ t \in \mu\}$ of the group A. We assume that Δ is fixed.

Now we can interpret the second order theory of the group A. We shall do the following.

In addition to the set $\mathbf{F}$, consisting of μ independent projections onto countably generated direct summands of B, we shall also consider a similar set $\mathbf{G}$ with the only additional condition that for every $g \in \mathbf{G}$ we shall select one fixed projection g_0 onto a countably generated direct summand g_0A of gA such that if $gA = g_0A \oplus A'$, then A' is also countably generated.

Let us fix some $g \in \mathbf{G}$ and consider a homomorphism h such that $h(g_0A) \in \langle a_i \rangle$ for some $i \in \omega$, and if $g_t \in \mathbf{G}_g \setminus \mathbf{G}_g^0$, then either $h(g_tA) \subset fA$ for some $f \in \mathbf{F}$ or $h(g) = 0$.

Let the image h under gA be finitely generated and the inverse image of every $f \in \mathbf{F}$ contain at most $p-1$ elements of $\mathbf{G}_g$.

Then every such h is mapped to an element from A as follows: if h on $\mathbf{G}_g \setminus \mathbf{G}_g^0$ is a finite subset $\mathcal{F}$ of $\{g_t^i \mid t \in \mu,\ i \in \omega\}$, and $h(g_t^0) = a_k$, then we obtain an element

$$\sum_{f_t^i \in \mathcal{F}} \alpha_{ti} c_{it} + b,$$

where α_{it} is the multiplicity of the inverse image of f_t^i and $b = \Lambda(a_k)$.

It is clear that, as above, for such a mapping h we can write $h \in \mathrm{End}_g$. Two elements $h_1, h_2 \in \mathrm{End}_g$ are said to be equivalent if there exists an automorphism α of the group gA substituting elements of $\mathbf{G}_g$ and replacing $g_0 \in \mathbf{G}_g^0$ such that $h_1\alpha$ and h_2 coincide. As usual, the set End_g factorized by such an equivalence is denoted by $\widetilde{\mathrm{End}}_g$. Addition on the set $\widetilde{\mathrm{End}}_g$ is obvious: the images of an element g_0 are added, the number of inverse images of every f_{it} is added, and, if it is greater than $p-1$, then we have an excess of p of inverse images of f_{it}, and we add one more inverse image of $f_{i-1,t}$ and we add the known $b_{it} = \beta(\Delta(f_{it}))$ to the image of g_0.

The rest of the proof is similar to the previous cases, because we have μ independent elements $g_t \in \mathbf{G}$ and for each of them we can interpret the theory $\mathrm{Th}(A)$.

8 The Main Theorem

We recall (see Sec. 4.2) that if $A = D \oplus G$, where D is divisible and G is reduced, then the *expressible rank of the group A* is the cardinal number

$$r_{\exp} = \mu = \max(\mu_D, \mu_G),$$

where μ_D is the rank of D, and μ_G is the rank of a basic subgroup of G.

Theorem 1. *For any infinite p-groups A_1 and A_2 elementary equivalence of endomorphism rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ implies coincidence of the second order theories $\mathrm{Th}_2^{r_{\exp}(A_1)}(A_1)$ and $\mathrm{Th}_2^{r_{\exp}(A_2)}(A_2)$ of the groups A_1 and A_2, bounded by the cardinal numbers $r_{\exp}(A_1)$ and $r_{\exp}(A_2)$, respectively.*

Proof. Since the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ are elementary equivalent, they satisfy the same first order sentences. If in the ring $\mathrm{End}(A_1)$ for some natural k the sentence

$$\forall x\,(p^k x = 0) \wedge \exists x\,(p^{k-1} x \neq 0)$$

holds, then the group A_1 is bounded and the maximum of the orders of its elements is equal to p^k. It is clear that in this case the same holds also for the group A_2, and the theorem follows from Proposition 1 (Sec. 5.4).

Now suppose that neither the group A_1, nor the group A_2 is bounded. Let for some natural k the sentence ψ_{p^k} (from Sec. 4.4) hold in the ring $\mathrm{End}(A_1)$.

Then this sentence holds also in the ring $\mathrm{End}(A_2)$, and therefore the groups A_1 and A_2 are direct sums $D_1 \oplus G_1$ and $D_2 \oplus G_2$, respectively, where the groups D_1 and D_2 are divisible and the groups G_1 and G_2 are bounded by the number p^k. Further, the sentence ψ_{p^k} fixes the projections ρ_D and ρ_G on the groups D and G, respectively. If $\rho_G = 0$, then the groups A_1 and A_2 are divisible and in this case the theorem follows from Proposition 2.

Let the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ satisfy the sentence

$$\begin{aligned}\tilde{\psi}^2_{p^k} := \exists \rho_D\, \exists \rho_G\, (\psi_{p^k}(\rho_D, \rho_G) \wedge \exists h\, (\rho_D h \rho_G = h\rho_G \\ \wedge \forall \rho_1\, \forall \rho_2\, (\mathrm{Idem}^*(\rho_1) \wedge \mathrm{Idem}^*(\rho_2) \wedge \rho_1\rho_G = \rho_1 \wedge \rho_2\rho_G = \rho_2 \wedge \rho_1\rho_2 = \rho_2\rho_1 = 0 \\ \Rightarrow \exists \rho_1'\, \exists \rho_2'\, (\mathrm{Idem}^*(\rho_1') \wedge \mathrm{Idem}^*(\rho_2') \wedge \rho_1'\rho_D = \rho_1' \wedge \rho_2'\rho_D = \rho_2' \\ \wedge \rho_1'\rho_2' = \rho_2'\rho_1' = 0 \wedge \rho_1' h \rho_1 = h\rho_1 \neq 0 \wedge \rho_2' h \rho_2 = h\rho_2 \neq 0)))).\end{aligned}$$

This sentence (in addition to the conditions of the sentence ψ_{p^k}) says that there exists an endomorphism h of the group A such that h maps G into D and any two independent cyclic summands $\rho_1 A$ and $\rho_2 A$ of the group G are mapped into independent quasicyclic summands $\rho_1' A$ and $\rho_2' A$ of D, i.e., there exists an embedding of G into the group D. This implies that $|G| \leq |D|$, i.e., if the sentence $\psi^2_{p^k}$ holds in $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$, then the groups A_1 and A_2 are isomorphic to direct sums $D_1 \oplus G_1$ and $D_2 \oplus G_2$, where $|D_1| \geq |G_1|$ and $|D_2| \geq |G_2|$. In this case, the theorem follows from Proposition 3.

If the sentence ψ_{p^k} holds in $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$, but the sentence $\psi^2_{p^k}$ is false, then the groups A_1 and A_2 are direct sums $D_1 \oplus G_1$ and $D_2 \oplus G_2$, where $|D_1| < |G_1|$ and $|D_2| < |G_2|$. In this case, the theorem follows from Proposition 4.

If for no natural k the sentence ψ_{p^k} belongs to the theory $\mathrm{Th}(\mathrm{End}(A_1))$, then for no natural k the sentence ψ_{p^k} belongs to the theory $\mathrm{Th}(\mathrm{End}(A_2))$, and therefore both groups A_1 and A_2 have unbounded basic subgroups.

Let us consider the formula

$$\begin{aligned}\psi(\rho_D, \rho_G) := \mathrm{Idem}(\rho_D) \wedge \mathrm{Idem}(\rho_G) \wedge (\rho_D\rho_G = \rho_G\rho_D = 0) \wedge (\rho_D + \rho_G = 1) \\ \wedge \forall x\, (\rho_D x \rho_D = 0 \vee p(\rho_D x \rho_D) \neq 0) \wedge \forall \rho'\, (\mathrm{Idem}^*(\rho') \wedge \rho'\rho_G = \rho' \Rightarrow \neg(\forall x\, (\rho' x \rho' = 0 \vee p(\rho' x \rho') \neq 0))).\end{aligned}$$

This formula says that A is the direct sum of its subgroups $\rho_D A$ and $\rho_G A$, the group $\rho_D A$ is divisible, and the group $\rho_G A$ is reduced.

Let us consider the sentence

$$\psi^2 := \exists \rho_D\, \exists \rho_G\, \exists h\, (\psi(\rho_D, \rho_G) \wedge \rho_D h \rho_G = h\rho_G \wedge \forall \rho\, (\mathrm{Idem}^*(\rho) \wedge \rho\rho_G = \rho \Rightarrow \forall c \in Z\, (c\rho \neq 0 \Rightarrow ch\rho \neq 0))).$$

This sentence says that A is the direct sum of a divisible subgroup $D = \rho_D A$ and a reduced subgroup $G = \rho_G A$, and there exists an embedding $h\colon G \to D$. Therefore $|G| \leq |D|$. If the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ satisfy the sentence ψ^2, then for the groups A_1 and A_2 we have $A_1 = D_1 \oplus G_1$, $A_2 = D_2 \oplus G_2$, $|D_1| \geq |G_1|$, $|D_2| \geq |G_2|$, and the theorem follows from Proposition 1.

Now suppose that the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ do not satisfy the sentence ψ^2. In this case, $A_1 = D_1 \oplus G_1$, $A_2 = D_2 \oplus G_2$, $|D_1| < |G_1|$, and $|D_2| < |G_2|$.

Recall the formulas from Sec. 7.2.

Let us consider the sentence

$$\begin{aligned}\psi^3 := \exists \rho_D\, \exists \rho_G\, (\psi(\rho_D, \rho_G) \wedge \neg\psi^2 \wedge \exists \varphi_B\, (\mathrm{Base}(\varphi_B) \wedge \forall \rho\, (\mathrm{Idem}^*(\rho) \Rightarrow \exists \rho'\, (\mathrm{Idem}^*(\rho') \wedge o(\rho') > o(\rho) \\ \wedge \exists f\, (\mathrm{Ord}_{\rho'}(f) \wedge \forall f'\, (\mathrm{Idem}^*(f') \wedge f'f = f' \Rightarrow \forall c \in Z\, (cf' \neq 0 \Rightarrow cf'\varphi_B \neq 0)) \\ \wedge \exists h\, (\forall f_1\, (\mathrm{Idem}^*(f_1) \wedge \forall c \in Z\, (cf_1 \neq 0 \Rightarrow cf_1\varphi_B \neq 0)\end{aligned}$$

$$\Rightarrow \exists f_2\,(\mathrm{Idem}^*(f_2) \wedge f_2 f = f_2 \wedge f_1 h = h f_2 = f_1 h f_2 \neq 0)))))))).$$

This sentence says that

1. $A = \rho_D A \oplus \rho_G A = D \oplus G$, where D is divisible, G is reduced, and $|D| < |G|$;
2. φ_B is an endomorphism with the image $\varphi_B(A)$ coinciding with some basic subgroup B;
3. for every natural k there exists a natural n such that in the group B there exists a direct summand (which is a sum of cyclic groups of order p^n) having the same power as the group B.

Therefore the sentence ψ^3 says that the final rank of a basic subgroup of the group G coincide with its rank. The sentence

$$\psi^4 := \exists \varphi_B\,(\mathrm{Base}(\varphi_B) \wedge \forall \rho\,(\mathrm{Idem}^*(\rho) \Rightarrow \forall f\,(\mathrm{Ord}_\rho(f) \Rightarrow \mathrm{Fin}(f) \vee \exists h\,(\forall f_1\,(\mathrm{Idem}^*(f_1)$$
$$\wedge \forall c \in Z\,(cf_1 \neq 0 \Rightarrow cf_1\varphi_B \neq 0) \Rightarrow \exists f_2\,(\mathrm{Idem}^*(f_2) \wedge f_2 f = f_2 \wedge f_1 h = h f_2 = f_1 h f_2 \neq 0))))))$$

means that in a basic subgroup B for every natural n every direct summand which is a direct sum of cyclic groups of order p^n either is finite or has the same power as the group B, so the group B is countable.

Therefore if the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ satisfy the sentence $\psi^3 \wedge \neg\psi^4$, then the groups A_1 and A_2 are direct sums $D_1 \oplus G_1$ and $D_2 \oplus G_2$, where $|D_1| < |G_1|$, $|D_1| < |G_2|$, and the final ranks of basic subgroups of A_1 and A_2 coincide with their ranks and are uncountable. In this case the theorem follows from Proposition 1. If the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ satisfy the sentence $\psi^3 \wedge \psi^4$, then their basic subgroups are countable and in this case the theorem follows from Proposition 4. Now we have only two cases, and to separate them we shall write the sentence

$$\psi^5 := \exists \varphi_B\,\exists \bar\rho\,(\mathrm{Base}(\varphi_B) \wedge \mathrm{Idem}^*(\bar\rho) \wedge \forall \rho\,(\mathrm{Idem}^*(\rho) \wedge o(\rho) > o(\bar\rho)$$
$$\Rightarrow \forall f\,(\mathrm{Ord}_\rho(f) \Rightarrow \mathrm{Fin}(f) \vee \exists h\,(\forall f_1(\mathrm{Idem}^*(f_1) \wedge o(f_1) > o(\rho)$$
$$\wedge \forall c \in Z\,(cf_1 \neq 0 \Rightarrow cf_1\varphi_B \neq 0) \Rightarrow \exists f_2\,(\mathrm{Idem}^*(f_2) \wedge f_2 f = f_2 \wedge f_1 h = h f_2 = f_1 h f_2 \neq 0)))))),$$

which means that there exists a number k such that in a basic subgroup B for every natural $n > k$, every direct summand which is a sum of cyclic groups of order p^n either is finite or has the same power as the direct summand of the group B generated by all generators of order greater than p^k.

Naturally, this means that the final rank of the group B is countable.

Now if the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ satisfy the sentence $\neg\psi^3 \wedge \neg\psi^5$, then the final ranks of basic subgroups of A_1 and A_2 are uncountable and do not coincide with their ranks. In this case, the theorem follows from Proposition 3.

If the rings $\mathrm{End}(A_1)$ and $\mathrm{End}(A_2)$ satisfy the sentence $\neg\psi^3 \wedge \psi^5$, then the final ranks of basic subgroups of A_1 and A_2 are countable and do not coincide with their ranks, and in this case the theorem follows from Sec. 7.6. □

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